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Coefficient Multipliers for the Privalov class in a Disk

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We obtain exact estimates of the growth and the Taylor coefficients of analytic functions from the Privalov classes in the unit disk. Also we describe coefficient multipliers from the Privalov classes into the Hardy classes.

Keywords: Privalov classes, Taylor coefficients, coefficient multiplier, maximal growth, analytic functions.

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Introduction

Let \mathbb{C} be the complex plane, D be the unit disk on \mathbb{C} , $H(D)$ be the set of all functions, holomorphic in D . For all $0 < q < +\infty$ we define the Privalov class of function Π_q as follows (see [8]):

$$\Pi_q = \left\{ f \in H(D) : \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\theta})|)^q d\theta < +\infty \right\}.$$

Note that the classes Π_q were first considered by I. I. Privalov in [8]. In the case $1 \leq q < +\infty$ the Privalov spaces were studied by M. Stoll, V. I. Gavrilo, A. V. Subbotin, D. A. Efimov, R. Mestrovic, Z. Pavicevic, etc. The monograph [2] contains a brief overview of their results. The case $0 < q < 1$ has been little studied in the scientific literature. In this paper we obtain exact estimates for the maximum modulus and the Taylor coefficients of functions from the classes Π_q ($0 < q < 1$) (Section 1), on this basis we describe coefficient multipliers from the Privalov classes Π_q ($0 < q < 1$) into the Hardy classes H^p ($0 < p < +\infty$) (Section 2).

Notice that the problem of describing the Taylor coefficients for analytic functions of the Nevanlinna class was first solved by S. N. Mergelyan in the early 20th century (see [9, p. 152]). Later on these questions in the Hardy classes were investigated by G. Hardy and D. Littlewood, A. A. Friedman (see [3]), in V. I. Smirnov's classes by N. Yanagihara [20], in the Privalov classes Π_q ($q > 1$) by M. Stoll [18], in the plane Nevanlinna classes by S. V. Shvedenko [17], in the weighted classes of analytic functions in a disk with restrictions on the Nevanlinna characteristic by F. A. Shamoyan and E. N. Shubabko [14], and by the author of this paper (see [10]).

As the authors observe in [2, p. 148], the notion of a coefficient multiplier arises naturally in the study of asymptotic properties of the Taylor coefficients for functions from certain classes. In a simplified form, the problem is posed as follows: which factors the Taylor coefficients of a function of a given class must be multiplied by for them to acquire special properties, for example, being bounded or form an absolutely convergent series. Requiring that the resulting

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products be Taylor coefficients of functions from some other class, we arrive to definition of the coefficient multiplier.

Let X and Y be some classes of functions analytic in the unit disk D .

Definition 1. The sequence of complex numbers $\Lambda = \{\lambda_k\}_{k=1}^{+\infty}$ is called a coefficient multiplier from class X into class Y if for any function $f \in X$, $f(z) = \sum_{k=0}^{+\infty} a_k z^k$, we have $\Lambda(f)(z) = \sum_{k=0}^{+\infty} \lambda_k a_k z^k \in Y$. It is denoted by $CM(X, Y)$.

Numerous works are devoted to the description of multipliers in various classes of holomorphic functions. We recall some of them: [1, 2, 4–6, 11, 13–16, 19], and etc.

1. On maximal growth and the Taylor coefficients for functions from the Privalov classes

The following statement is valid:

Theorem 1.1. If $f \in \Pi_q$ then

$$\ln^+ M(r, f) = o((1-r)^{-1/q}), \quad r \rightarrow 1-0, \quad (1)$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Remark. Throughout the paper we study the Privalov class Π_q with the parameter $0 < q < 1$. Also, unless otherwise stated, we denote by $c, c_1, \dots, c_n(\alpha, \beta, \dots)$ arbitrary positive constants depending on α, β, \dots , whose specific values are immaterial.

Proof. We choose an arbitrary point $z_0 \in D$ and by definition put $K_{z_0} = \{\zeta \in D : |\zeta - z_0| < \frac{1}{2}(1 - |z_0|)\}$. Let dm_2 be the planar Lebesgue measure. From the inequality (see [7, p. 144], Theorem 9.1.1, equation(9.3)) which holds for all $0 < q < 1$:

$$(\ln^+ |f(z_0)|)^q \leq \frac{c_q}{(1 - |z_0|)^2} \int_{K_{z_0}} (\ln^+ |f(\zeta)|)^q dm_2(\zeta),$$

we obtain

$$(\ln^+ |f(z_0)|)^q \leq \frac{c_q}{(1 - |z_0|)^2} \int_{|z_0| - \frac{1-|z_0|}{2}}^{|z_0| + \frac{1-|z_0|}{2}} \int_{-\pi}^{\pi} (\ln^+ |f(\rho e^{i\theta})|)^q d\theta d\rho,$$

whence we have:

$$\begin{aligned} (\ln^+ |f(z_0)|)^q &\leq \frac{c_q}{(1 - |z_0|)} \sup_{|z_0| - \frac{1-|z_0|}{2} < \rho < |z_0| + \frac{1-|z_0|}{2}} \int_{-\pi}^{\pi} (\ln^+ |f(\rho e^{i\theta})|)^q d\theta \leq \\ &\leq \frac{c_q}{(1 - |z_0|)} \sup_{0 < \rho < 1} \int_{-\pi}^{\pi} (\ln^+ |f(\rho e^{i\theta})|)^q d\theta. \end{aligned}$$

Now the required estimate (1) follows. \square

Theorem 1.2. If $\sum_{k=0}^{+\infty} a_k z^k$ is the Taylor series expansion for a function $f \in \Pi_q$, then

$$\ln^+ |a_k| = o\left(k^{\frac{1}{1+q}}\right), k \rightarrow +\infty. \quad (2)$$

Proof. We prove this theorem using the method of S.N. Mergelyan (see [9, p. 152]). From the Cauchy inequality and the estimate (1) of Theorem 1.1 it follows that for any arbitrarily small $\varepsilon > 0$ there exists $r_\varepsilon \in (0, 1)$ such that

$$|a_k| \leq r^{-k} \exp\left\{\varepsilon(1-r)^{-\frac{1}{q}}\right\}, r_\varepsilon < r < 1, n = 0, 1, \dots, \quad (3)$$

which is equivalent to

$$\ln^+ |a_k| \leq \varepsilon(1-r)^{-\frac{1}{q}} - k \ln r, r_\varepsilon < r < 1, n = 0, 1, \dots. \quad (4)$$

Introduce the function

$$\phi(r) = \varepsilon(1-r)^{-\frac{1}{q}} - k \ln r.$$

We investigate it to find its exact lower bound. Calculate the derivative:

$$\phi'(r) = \frac{\varepsilon}{q} \cdot \frac{1}{(1-r)^{\frac{1}{q}+1}} - \frac{k}{r}.$$

We find the minimum of the function $\phi(r)$, solving the equation $\phi'(r) = 0$:

$$\frac{\varepsilon}{q} \cdot \frac{r}{(1-r)^{\frac{1}{q}+1}} = k. \quad (5)$$

The solution of this equation exists and unique on the interval $(0, 1)$.

For convenience, we introduce the following notation:

$$c_k = \frac{1}{\delta \sqrt{r_k}}, \quad s_k = \frac{1 - r_k}{\delta \sqrt{r_k}},$$

where $\delta > 1$.

We can assume that $s_k < c_k \leq 1$. Indeed, the inequality $s_k < c_k$ is obvious.

Now, $c_k \leq 1$ is equivalent to

$$\sqrt{r_k} \geq \frac{1}{\delta}, \quad (6)$$

while $s_k < 1$ is equivalent to

$$\sqrt{r_k} > \frac{\sqrt{\delta^2 + 4} - \delta}{2}, \quad (7)$$

and (6) follows from (7).

In the new notation equation (5) takes the form:

$$\frac{\varepsilon}{q\delta^2} \cdot \frac{1}{s_k^2} \cdot \left(\frac{c_k}{s_k}\right)^{\frac{1}{q}-1} = k$$

or

$$\frac{s_k^{\frac{1}{q}+1}}{c_k^{\frac{1}{q}-1}} = \frac{\varepsilon}{kq\delta^2}.$$

Since $c_k \leq 1$, the last equality implies the estimate:

$$s_k \leq \left(\frac{\varepsilon}{kq\delta^2}\right)^{\frac{q}{q+1}}. \quad (8)$$

From the same equation we obtain:

$$\left(\frac{c_k}{s_k}\right)^{\frac{1}{q}} = \left(\frac{ks_k^2 q \delta^2}{\varepsilon}\right)^{\frac{1}{1-q}}.$$

Taking into account the estimate (8), we have:

$$\left(\frac{c_k}{s_k}\right)^{\frac{1}{q}} \leq \left(\frac{q\delta^2}{\varepsilon}\right)^{\frac{1}{1+q}} \cdot k^{\frac{1}{1+q}}. \quad (9)$$

Using (8), (9), we estimate the value of the function $\phi(r)$ at the minimum point $r = r_k$:

$$\phi(r_k) = \varepsilon(1 - r_k)^{-\frac{1}{q}} - k \ln r_k.$$

The application of (9) yields:

$$\phi(r_k) \leq \varepsilon \left(\frac{q\delta^2}{\varepsilon}\right)^{\frac{1}{1+q}} \cdot k^{\frac{1}{1+q}} - k \ln r_k.$$

To estimate the last term we note that

$$\frac{(r_k)^{-\frac{1}{2}} - r_k^{\frac{1}{2}}}{2} = \frac{\exp(-\frac{1}{2} \ln r_k) - \exp(\frac{1}{2} \ln r_k)}{2} = -\operatorname{sh}\left(\frac{1}{2} \ln r_k\right) = \operatorname{sh}\left(-\frac{1}{2} \ln r_k\right) = \frac{s_k \delta}{2},$$

whence

$$\begin{aligned} -\ln r_k &= 2 \operatorname{arcsch} \frac{s_k \delta}{2} \leq 2 \frac{s_k \delta}{2}, \\ -k \ln r_k &\leq k s_k \delta. \end{aligned}$$

Thus we have:

$$\phi(r_k) \leq k^{\frac{1}{1+q}} \varepsilon^{\frac{q}{1+q}} (q\delta^2)^{\frac{1}{1+q}} \cdot \left(1 + \frac{1}{q\delta}\right). \quad (10)$$

The required estimate (2) follows. \square

2. Description of the coefficient multipliers from the Privalov classes into the Hardy classes

Theorem 2.1. *Let $0 < p \leq +\infty$, $\Lambda = \{\lambda_k\}_{k=1}^{+\infty} \subset \mathbb{C}$. For $\Lambda = CM(\Pi_q, H^p)$ it is necessary and sufficient to have*

$$|\lambda_k| = O\left(\exp\left(-c \cdot k^{\frac{1}{q+1}}\right)\right), \quad k \rightarrow +\infty, \quad (11)$$

for some $c > 0$.

The proof of this theorem is based on auxiliary statements.

Lemma 2.2. *(see [2, Lemma 9.7]) Let F and H be linear classes of holomorphic functions in the unit disk D with metrics, convergence in which is not weaker than the uniform convergence on compact subsets of D . Then each coefficient multiplier from class F into class H is a linear and closed operator.*

To formulate the next Lemma we introduce a metric on the class Π_q :

$$\rho(f, g) = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^q (1 + |f(re^{i\theta}) - g(re^{i\theta})|) d\theta. \quad (12)$$

Lemma 2.3. *The class Π_q with respect to the metric (12) is an F -space.*

Proof. The proof of this statement is equivalent to establishing the properties a)–d) of a metric (see [12]):

- a) $\rho(f, g) = \rho(f - g, 0)$ is obvious.
- b) Π_q is a complete metric space.

Let $\{f_n\}$ be an arbitrary fundamental (Cauchy) sequence from the class Π_q , i.e. $\forall \varepsilon > 0 \exists N(\varepsilon) > 0 : \forall n, m > N \Rightarrow \rho(f_n, f_m) < \varepsilon$. We show that it converges to some function $f \in \Pi_q$. Note that functions $\ln(1 + |f_n|)$ are subharmonic in D . In the same way as in the proof of Theorem 1.1 we use the equation from Theorem 9.1.1 of [7, p. 144] to obtain:

$$\begin{aligned} \ln^q(1 + |f_n(Re^{i\varphi}) - f_m(Re^{i\varphi})|) &\leq \frac{c_q}{(1-R)^2} \sup_{0 < r < 1} \int_{-\pi}^{\pi} \ln^q(1 + |f_n(re^{i\theta}) - f_m(re^{i\theta})|) d\theta, \\ \ln^q(1 + |f_n(Re^{i\varphi}) - f_m(Re^{i\varphi})|) &\leq \frac{2\pi c_q}{(1-R)^2} \rho(f_n, f_m), \end{aligned}$$

whence we have

$$|f_n(Re^{i\varphi}) - f_m(Re^{i\varphi})| \rightarrow 0, \quad n, m \rightarrow +\infty,$$

for all $0 < R < 1$, $\varphi \in [-\pi, \pi]$. So the sequence $\{f_n\}$ converges uniformly in the unit disk for some function $f \in H(D)$. Now we prove that $f \in \Pi_q$.

$$\begin{aligned} \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\theta})|)^q d\theta &\leq \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(1 + |f(re^{i\theta})|)^q d\theta \leq \\ &\leq \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^q(1 + |f(re^{i\theta}) - f_n(re^{i\theta})| + |f_n(re^{i\theta})|) d\theta. \end{aligned}$$

Since $(a+b)^q \leq (a^q + b^q)$ for any $a > 0, b > 0, 0 < q < 1$, we have

$$\begin{aligned} \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\theta})|)^q d\theta &\leq \\ &\leq \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^q(1 + |f(re^{i\theta}) - f_n(re^{i\theta})|) + \ln^q(1 + |f_n(re^{i\theta})|)) d\theta \leq const. \end{aligned}$$

We conclude that Π_q is a complete metric space.

c) If $f, f_n \in \Pi_q$ and $\rho(f_n, f) \rightarrow 0, n \rightarrow +\infty$, then for any $\beta \in \mathbb{C}$ we have $\rho(\beta f_n, \beta f) \rightarrow 0, n \rightarrow +\infty$.

For $|\beta| < 1$ the property immediately follows. Assume that $|\beta| > 1$, without loss of generality we may take $\beta > 1$. Since the sequence $\{f_n\}$ converges, it is fundamental (Cauchy). As stated above, from that follows its uniform convergence inside D .

Since for any $\beta \geq 1$ and $x \geq 0$ the estimate $(1 + \beta x) \leq (1 + x)^\beta$ is valid, we have

$$\begin{aligned} \rho(\beta f_n, \beta f) &= \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^q(1 + \beta |f_n(re^{i\theta}) - f(re^{i\theta})|) d\theta \leq \\ &\leq \sup_{0 < r < 1} \frac{\beta^q}{2\pi} \int_{-\pi}^{\pi} \ln^q(1 + |f_n(re^{i\theta}) - f(re^{i\theta})|) d\theta = \beta^q \rho(f_n, f), \end{aligned}$$

whence the property c) follows.

d) If $\beta_n, \beta \in \mathbb{C}$ and $\beta_n \rightarrow \beta$, then we have $\rho(\beta_n f, \beta f) \rightarrow 0, n \rightarrow +\infty$ for any function $f \in \Pi_q$. The property immediately follows from the inequality

$$\ln(1 + |\beta_n - \beta||f|) \leq \ln(1 + |f|) + \ln(1 + |\beta_n - \beta|).$$

Lemma 2.3 is proved. □

Lemma 2.4. *Let the sequence of complex numbers $\{\lambda_k\}_{k=1}^{+\infty}$ satisfy the condition:*

$$|\lambda_k| = O\left(\exp\left(-c_k \cdot k^{\frac{1}{q+1}}\right)\right), \quad k \rightarrow +\infty \quad (13)$$

for an arbitrary positive sequence $\{c_k\}_{k=1}^{+\infty}$, $c_k \downarrow 0$, $k \rightarrow +\infty$. Then there exists $c > 0$ such that for all $k \in \mathbb{N}$ the condition (11) is true.

The proof of Lemma 2.4 repeats the arguments of the article by N. Yanagihara [19] (see Lemma 1) with the exponent $\frac{1}{q+1}$.

Lemma 2.5. *Let $0 < q < 1$,*

$$g(z) = \exp \frac{c}{(1-z)^{\frac{1}{q}}}, \quad z \in D, \quad (14)$$

where $0 < c < \frac{1}{q}$, $\sum_{n=1}^{+\infty} a_n(c)z^n$ be the Taylor series expansion for g . Then the following estimate is valid:

$$|a_n(c)| \geq \exp\left(c^{\frac{q}{q+1}} \cdot n^{\frac{1}{q+1}}\right). \quad (15)$$

The proof of Lemma 2.5 repeats the arguments presented in the author's thesis (see [11, p. 104], Lemma 2.7) with the exponent of $\frac{1}{q}$. The method goes back to Mergelyan S. N. (see [9]).

As stated above, from $\rho(f_n, f) \rightarrow 0$, $n \rightarrow +\infty$ it follows that the sequence $f_n(z)$ uniformly converges to $f(z)$ in D . Therefore if $f_n(z) = \sum_{k=0}^{+\infty} a_k^{(n)} z^k$ and $f(z) = \sum_{k=0}^{+\infty} a_k z^k$, then $a_k^{(n)} \rightarrow a_k$, $n \rightarrow +\infty$.

Let X be an F-space consisting of complex sequences $\{b_k\}_k$ such that convergence of a sequence $\beta^{(n)} = \{b_k^{(n)}\}$ to $\beta = \{b_k\}$, $n \rightarrow +\infty$ implies coordinate-wise convergence $b_k^{(n)} \rightarrow b_k$, $n \rightarrow +\infty$, $k = 0, 1, 2, \dots$.

Consider a coefficient multiplier $\Lambda = CM(\Pi_q, X)$. By Lemma 2.2, Λ is a closed operator, therefore by the closed graph theorem (see [12]) Λ is a continuous operator and it maps bounded sets in the class Π_q into bounded sets in X .

Proof of Theorem 2.1. Let $\Lambda = \{\lambda_k\}_{k=1}^{+\infty}$ be a coefficient multiplier from the class Π_q into the class X . We prove that there exists $c > 0$ such that the estimate (11) is valid, i.e.

$$|\lambda_k| = O\left(\exp\left(-c \cdot k^{\frac{1}{q+1}}\right)\right), \quad k \rightarrow +\infty.$$

According to Lemma 2.4, it is enough to show that Λ satisfies the estimate (13) for some positive infinitesimal sequence $\{c_k\}$.

We choose the sequence $\{c_k\}$ so that the following estimates are valid:

$$k^{-\frac{1}{2q}} \leq c_k \leq \frac{1}{2}. \quad (16)$$

We consider in Π_q the sequence of functions

$$f_k(z) = g(r_k z) = \exp \frac{c_k}{(1-r_k z)^{\frac{1}{q}}}, \quad k = 1, 2, \dots, \quad 0 < r_k < 1, \quad (17)$$

satisfying the conditions of Lemma 2.5, and the double inequality

$$\sqrt{1 - \frac{\pi^2}{k-1}} \leq r_k \leq \sqrt{1 - \frac{\pi^2}{B_k-1}}, \quad B_k = \pi \left(\frac{\gamma_k}{c_k}\right)^q, \quad B_k > 1, \quad (18)$$

where γ_k is a positive infinitesimal sequence such that $\frac{c_k}{\gamma_k} \leq k^{-\frac{1}{q}}$, $k = 1, 2, \dots$.

It is obvious that $r_k \rightarrow 1 - 0$, $k \rightarrow +\infty$.

Let us show that $f_k \in \Pi_q$.

$$\begin{aligned} & \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |f_k(re^{i\theta})|)^q d\theta = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ \left| \exp \frac{c_k}{(1 - r_k r e^{i\theta})^{\frac{1}{q}}} \right| \right)^q d\theta \leq \\ & \leq \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{c_k^q}{|1 - r_k r e^{i\theta}|} d\theta = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{c_k^q}{\sqrt{1 - 2r_k r \cos \theta + (r_k r)^2}} d\theta = \\ & = \sup_{0 < r < 1} \frac{1}{\pi} \int_0^{\pi} \frac{c_k^q}{\sqrt{(1 - r_k r)^2 + 4r_k r \sin^2(\theta/2)}} d\theta = \sup_{0 < r < 1} \frac{c_k^q}{\pi} \left(\int_0^{1-r_k r} \dots + \int_{1-r_k r}^{\pi} \dots \right) \leq \\ & \leq \sup_{0 < r < 1} \frac{c_k^q}{\pi} \left(\int_0^{1-r_k r} \frac{1}{(1 - r_k r)} d\theta + \int_{1-r_k r}^{\pi} \frac{1}{\sqrt{(1 - r_k r)^2 (\frac{\theta}{\pi})^2 + 4r_k r (\frac{\theta}{\pi})^2}} d\theta \right) = \\ & = \sup_{0 < r < 1} \frac{c_k^q}{\pi} \left(1 + \int_{1-r_k r}^{\pi} \frac{d\theta}{\sqrt{(1 + r_k r)^2 (\frac{\theta}{\pi})^2}} \right) = \sup_{0 < r < 1} \frac{c_k^q}{\pi} \left(1 + \frac{\pi}{(1 + r_k r)} \ln \frac{\pi}{1 - r_k r} \right). \end{aligned}$$

Taking into account the evident inequality $\ln x < x$, $\forall x > 0$, and the condition (18) we conclude

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln^+ |f_k(re^{i\theta})|)^q d\theta \leq \sup_{0 < r < 1} \frac{c_k^q}{\pi} \left(1 + \frac{\pi^2}{(1 - (r_k r)^2)} \right) = \frac{c_k^q}{\pi} \left(1 + \frac{\pi^2}{(1 - r_k^2)} \right) \leq \gamma_k^q.$$

We prove that $\{f_k\}$ is a bounded sequence in the class Π_q , i.e. we show that there exists a real number $0 < \lambda < 1$ such that for all natural numbers k the inequality $\rho(\lambda f_k, 0) < \varepsilon$ is valid, where ε is a fixed positive number (see [12, p. 31]). First, we prove that

$$\ln(1 + |\lambda||g|) \leq (\ln(1 + |\lambda|) + \ln^+ |g|). \quad (19)$$

Indeed, if $|g| \leq 1$ then $|\lambda||g| \leq |\lambda|$, and the estimate (19) follows immediately.

If $|g| \geq 1$ then $\ln(1 + |\lambda||g|) \leq \ln(|g| + |\lambda||g|) \leq \ln(1 + |\lambda|) + \ln^+ |g|$.

Now we prove the inequality $\rho(\lambda f_k, 0) < \varepsilon$. Since $(a + b)^q \leq (a^q + b^q)$ for any $a > 0, b > 0$, $0 < q < 1$, using the estimate (19) we get

$$\rho(\lambda f_k, 0) = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^q (1 + |\lambda f_k(re^{i\theta})|) d\theta \leq \ln^q(1 + |\lambda|) + (\gamma_k)^q.$$

Since $\gamma_k = o(1)$, $k \rightarrow +\infty$, $\forall \varepsilon > 0 \exists k_0 \in \mathbb{N} : \forall k \geq k_0$ the following inequality holds: $\gamma_k < \sqrt[q]{\frac{\varepsilon}{2}}$.

Choosing λ_{k_0} so that $\ln(1 + |\lambda_{k_0}|) < \sqrt[q]{\frac{\varepsilon}{2}}$, we see that starting from some number k_0 all elements of the sequence $\{f_k\}$ are contained in a ball of radius ε .

Since Π_q is an F -space, for all numbers $k < k_0$ there exists a positive number λ_k such that $\forall \lambda \in \mathbb{C}$, $|\lambda| \leq \lambda_k$ the following inequality is valid: $\rho(\lambda f_k, 0) < \varepsilon$. Assuming $\lambda_0 = \min(\lambda_1, \lambda_2, \dots, \lambda_{k_0})$, we obtain that for all $|\lambda| \leq \lambda_0$ the sequence $\{f_k\}$ is contained in a ball of radius ε , i.e. $\rho(\lambda f_k, 0) < \varepsilon$.

Owing to arbitrariness of the choice of ε , we conclude that $\{f_k\}$ is a bounded sequence in the class Π_q .

Since the sequence $\{f_k\}$ is bounded in Π_q , we get that the coefficient multiplier $\Lambda(f_k)$ is bounded in X .

Let $X = H^p$. We have

$$\|\Lambda(f_k)\|_{H^p} \leq C, \quad C > 0.$$

Fix a number $k \in \mathbb{N}$. If $f_k(z) = \sum_{n=0}^{+\infty} a_n^{(k)} z^n \in \Pi_q$, then $\Lambda(f_k)(z) = \sum_{n=0}^{+\infty} \lambda_n a_n^{(k)} z^n \in X$, therefore (see [3, c. 98])

$$|\lambda_n a_n^{(k)}| \leq c_p \|\Lambda(f_k)\|_{H^p} \cdot n^{\frac{1}{p}-1}, \quad \text{for } 0 < p < 1,$$

$$|\lambda_n a_n^{(k)}| \leq c_p \|\Lambda(f_k)\|_{H^p}, \quad \text{for } 1 \leq p \leq \infty,$$

whence we have

$$|\lambda_n a_n^{(k)}| \leq C \cdot c_p \cdot n^{\frac{1}{p}-1}, \quad \text{for } 0 < p < 1, \quad (20)$$

$$|\lambda_n a_n^{(k)}| \leq C \cdot c_p, \quad \text{for } 1 \leq p \leq \infty, \quad (21)$$

where c_p is a positive constant depending on p .

Since $f_k(z) = g(r_k z)$, we have $a_n^{(k)} = a_n(c_k) r_k^n$. According to Lemma 2.5,

$$|a_n^{(k)}| \geq r_k^n \exp\left(c_k^{\frac{q}{q+1}} n^{\frac{1}{q+1}}\right).$$

Using the inequality (18), we have:

$$|a_k^{(k)}| \geq \left(1 - \frac{\pi^2}{k-1}\right)^{\frac{k}{2}} \exp\left(c_k^{\frac{q}{q+1}} k^{\frac{1}{q+1}}\right). \quad (22)$$

From (20), (22) we obtain the following estimate for $0 < p < 1$:

$$|\lambda_k| \leq C \cdot c'_p \cdot \left(1 - \frac{\pi^2}{k-1}\right)^{-\frac{k}{2}} \cdot k^{\frac{1}{p}-1} \cdot \exp\left(-c_k^{\frac{q}{q+1}} k^{\frac{1}{q+1}}\right)$$

and applying now the estimate (16) we have:

$$|\lambda_k| \leq \tilde{C} \exp\left(-c_k^{\frac{q}{q+1}} k^{\frac{1}{q+1}}\right). \quad (23)$$

From the inequality (23), applying Lemma 2.4, we conclude that the estimate (11) holds. Analogously, for $1 \leq p < +\infty$ from (21) and (23) we obtain the required estimate.

We prove the converse statement of Theorem 2.1. Let a sequence $\Lambda = \{\lambda_k\}$ satisfy the condition (11) of Theorem 2.1 and $f \in \Pi_q$, $f(z) = \sum_{k=0}^{+\infty} a_k z^k$. From Theorem 1.2 it follows that

$$|a_k| \leq C_1 \exp\left(\varepsilon_k k^{\frac{1}{q+1}}\right), \quad \varepsilon_k \downarrow 0.$$

Choosing a number k_0 such that $\varepsilon_k < \frac{c}{2}$ for all $k \geq k_0$, we obtain:

$$|\lambda_k a_k| \leq C_2 \exp\left(-\frac{c}{2} k^{\frac{1}{q+1}}\right).$$

Since the series $\sum_{k=0}^{+\infty} \exp\left(-\frac{c}{2} k^{\frac{1}{q+1}}\right)$ converges, $\Lambda(f)(z) \in X$ for any specified choice of class X . Theorem 2.1 is proved. \square

Remark. Note that the method of the proof of Theorem 2.1 goes back to N. Yanagihara's work [19].

The immediate consequence of Theorem 2.1 is the statement that the estimates of Theorems 1.2 and 1.1 are unimprovable. The proof of this statement goes in the same manner as in the R. Meshtrovic's article (see [2, p. 152], Consequences 9.24, 9.26).

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О коэффициентных мультипликаторах классов Привалова в круге

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В статье получены точные оценки роста и коэффициентов разложения в ряд Тейлора функций из классов И. И. Привалова, полностью описаны коэффициентные мультипликаторы из класса Привалова в классы Харди.

Ключевые слова: класс Привалова, коэффициенты Тейлора, коэффициентный мультипликатор, рост, аналитические функции.