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Prym Differentials and Teichmüller Spaces

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In this article we study multiplicative meromorphic functions and differentials on Riemann surfaces of finite type. We proved an analogue of P. Appell's formula on decomposition of multiplicative functions with poles of arbitrary multiplicity into a sum of elementary Prym integrals. We construct explicit bases for some important factor spaces and prove a theorem on a fiber isomorphism of vector bundles and $n!$-sheeted mappings over Teichüller spaces. This theorem gives an important relation between spaces of Prym differentials on a compact Riemann surfaces and on a Riemann surfaces of finite type.

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Introduction

In the paper we study multiplicative meromorphic functions and differentials on Riemann surfaces of type (g, n). Recently the interest in this subject has increased in relation with applications in theoretical physics, in particular, in description of vortex-like patterns in ferromagnetics [1].

In this paper we continue constructing the general theory of functions on Riemann surfaces of the type (g, n) for multiplicative meromorphic function and differentials. We prove an analog of P. Appell's formula about the expansion of a multiplicative function with poles of arbitrary multiplicity into a sum of elementary Prym integrals. Also we construct explicit bases for some important quotient spaces and prove a theorem about fiber isomorphism of vector bundles and $n!-$sheeted mappings over Teichmüller spaces. This theorem gives an important relation between spaces of Prym differentials (abelian differentials) on compact Riemann surfaces and Riemann surfaces of finite type.

1. Preliminaries

Let F be a smooth compact oriented surface of genus $g \geqslant 2$, with the marking $\{a_k, b_k\}_{k=1}^g$, i.e. an ordered collection of standard generators of $\pi_1(F)$, and F_0 be a compact Riemann surface with the fixed complex-analytic structure on F. Fix different points $P_1, \ldots, P_n \in F$. We assign type (g, n) to a surface $F' = F \backslash \{P_1, \ldots, P_n\}$. By Γ' we denote the Fuchsian group of genus 1 acting invariantly in the disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and uniformizing the surface F_0'. Thus, $F_0' = U/\Gamma'$, where Γ' has the representation [2]

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$$\Gamma' = \langle A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n : \prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} C_1 \dots C_n = I \rangle.$$

Any other complex analytic structure on F' is given by a Beltrami differential μ on F_0', i. e. by an expression of the form $\mu(z)d\overline{z}/dz$, invariant with respect to the choice of the local parameter on F_0', where $\mu(z)$is a complex-valued function on F_0' and $\|\mu\|_{L_\infty(F_0')} < 1$. We denote this structure on F' by F_μ'.

Since the mapping $U \to F_0' = U/\Gamma'$ is a local diffeomorphism, any Beltrami differential μ on F_0' lifts to a Beltrami Γ'-differential μ on U, i. e. $\mu \in L_\infty(U)$, $\|\mu\|_\infty = \mathrm{esssup}_{z\in U}|\mu(z)| < 1$, and $\mu(T(z))\overline{T'(z)}/T'(z) = \mu(z), z \in U, T \in \Gamma'$, see [3].

In the work [3, p. 99] there were constructed abelian differentials $\zeta_1[\mu], \dots, \zeta_g[\mu]$ on $F_{[\mu]}$, that form a canonical base dual to a canonical homotopy base $\{a_k^\mu, b_k^\mu\}_{k=1}^g$ on F_μ, which depends holomorphically on moduli $[\mu]$ for a class of conformal equivalency of a marked Riemann surface F_μ. Further on, for brevity we shall write simply F_μ for the class of equivalence $F_{[\mu]}$. Here we assume that the class $[\mu]$ has Bers coordinates $h_1, h_2, \dots h_{3g-3}$ when embedding the Teichmüller space $\mathbb{T}_g(F_0)$ of compact Riemann surfaces into $\mathbb{C}^{3g-3}$. Moreover, the matrix of b-periods $\Omega(\mu) = (\pi_{jk}[\mu])_{j,k=1}^g$ on F_μ consists of complex numbers $\pi_{jk}[\mu] = \int_{\xi}^{B_k^\mu(\xi)} \zeta_j([\mu], w)dw$, $\xi \in w^\mu(U)$, and depends holomorphically on $[\mu]$.

For any fixed $[\mu] \in \mathbb{T}_g$ and $\xi_0 \in w^\mu(U)$ define a classical Jacobi mapping $\varphi : w^\mu(U) \to \mathbb{C}^g$ by the rule: $\varphi_j(\xi) = \int_{\xi_0}^{\xi} \zeta_j([\mu], w)dw$, $j = 1, \dots, g$. The quotient space $J(F) = \mathbb{C}^g/L(F)$ is called the marked Jacobi manifold for $F = F_0$, where $L(F)$ is a lattice over $\mathbb{Z}$, generated by the columns $e^{(1)}, \dots, e^{(g)}, \pi^{(1)}, \dots, \pi^{(g)}$ of the matrix (I_g, Ω), where I_g is an identity matrix of order g. The universal Jacobi manifold of order g is a fibered space over $\mathbb{T}_g$, with fiber over $[\mu] \in \mathbb{T}_g$ being a marked Jacobi manifold $J(F_\mu)$ for a marked Riemann surface F_μ [4].

A *character* ρ for F_μ' is any homomorphism $\rho : (\pi_1(F_\mu'), \cdot) \to (\mathbb{C}^*, \cdot)$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Further on we shall assume that $\rho(\gamma_j^\mu) = 1$, where γ_j^μ is a simple loop around only one puncture P_j on F_μ', $j = 1, \dots, n$.

Definition 1. *A multiplicative function f on F_μ' for the character ρ is a mermorphic function f on $w^\mu(U)$ such that $f(Tz) = \rho(T)f(z)$, $z \in w^\mu(U)$, $T \in \Gamma_\mu'$.*

Definition 2. *A Prym q-differential with respect to a Fuchsian group Γ' for ρ, or a (ρ, q)-differential, is a differential $\omega(z)dz^q$ such that $\omega(Tz)(T'z)^q = \rho(T)\omega(z)$, $z \in U$, $T \in \Gamma'$, $\rho : \Gamma' \to \mathbb{C}^*$.*

If a multipicative function f_0 on F_μ for ρ does not have zeroes or poles, then the character ρ is called *non-essential* and f_0 is called a *unit*. The characters which are not non-essential are called *essential* on $\pi_1(F_\mu)$. The set L_g of non-essential characters form a subgroup in the group $Hom(\Gamma_\mu, \mathbb{C}^*)$ of all characters on Γ_μ. A *divisor* on F_μ is a formal product $D = P_1^{n_1} \dots P_k^{n_k}$, $P_j \in F_\mu, n_j \in \mathbb{Z}$, $j = 1, \dots, k$.

Theorem (Abel's theorem for characters [2,5])**.** *Let D be a divisor on a marked variable compact Riemann surface $[F_\mu, \{a_1^\mu, \dots, a_g^\mu, b_1^\mu, \dots, b_g^\mu\}]$ of genus $g \geqslant 1$, and ρ be a character on $\pi_1(F_\mu)$. Then D is a divisor of a multiplicative function f on F_μ for ρ if and only if $degD = 0$ and*

$$\varphi(D) = \frac{1}{2\pi i}\sum_{j=1}^{g} \log \rho(b_j^\mu)e^{(j)} - \frac{1}{2\pi i}\sum_{j=1}^{g} \log \rho(a_j^\mu)\pi^{(j)}[\mu]\big(\equiv \psi(\rho, [\mu])\big),$$

where $\varphi[\mu] : F_\mu \to J(F_\mu)$ *is the Jacobi mapping.*

The class $M_1(\rho)$ consists of those Prym differentials for ρ on F'_μ, which have finitely many poles on F'_μ and admit meromorphic continuation to F_μ.

2. An analog of Appel's decomposition formula for a multiplicative function on a variable Riemann surface of finite type

Denote by $T^{(1)}_{\rho;Q} = -\int\limits_{Q_0}^{P} \tau^{(2)}_{\rho;Q}$ an elementary Prym integral of second kind on F_μ for an essential character ρ with only simple pole at Q with residue that depends holomorphically on $[\mu]$ and ρ, where $\tau^{(2)}_{\rho;Q}$ has zero residue at Q [5,6].

Theorem 1. *Let f be a branch of a function of class M_1 for an essential character ρ on a variable Riemann surface F'_μ of type (g, n), $g \geqslant 2$, $n > 0$, with pairwise distinct poles at $P_{n+1}, \ldots, P_{n+s}$ of multiplicities $q_{n+1}, \ldots, q_{n+s}$ with given principal parts:*

$$\frac{A_{j,q_j}}{(z - z(P_j))^{q_j}} + \ldots + \frac{A_{j1}}{(z - z(P_j))}, \quad j = n + 1, \ldots, n + s. \tag{1}$$

Then for an analytic continuation of f we have $(f) \geqslant \dfrac{1}{P_1^{q_1} \ldots P_{n+s}^{q_{n+s}}}$, $q_j \geqslant 0, j = 1, \ldots, n,$ *on* F_μ *and*

$$f = \sum_{j=1}^{n+s} \sum_{m=1}^{q_j} \left[\frac{A_{j,m}}{(m-1)!} \frac{\partial^{m-1} T^{(1)}_{\rho;P_j}}{\partial P_j^{m-1}} \right] + \sum_{j=1}^{g-1} \widetilde{c}_j \int_{Q_0}^{P} \tilde{\zeta}_j,$$

where $f = \dfrac{A_{j,q_j}}{(z - z(P_j))^{q_j}} + \cdots + \dfrac{A_{j,2}}{(z - z(P_j))^2} + \dfrac{A_{j,1}}{z - z(P_j)} + O(1)$ *for some branch in a neighborhood of* P_j, $j = 1, \ldots, n + s$, *на* F_μ, *and all summands depend holomorphically on* $[\mu]$ *and* ρ.

Let now ρ be a non-essential character. The proof of the previous expansion formula for an essential character does not work since in this case there is no Prym integral of second kind with only simple pole on F_μ. Therefore we need a Prym differential $\tau_{\rho;Q_1^2 Q_2^2}$ of second kind for a non-essential character ρ with two poles of second order at two distinct points Q_1 and Q_2 on $\triangle_\mu$ with zero residues at Q_1 and Q_2 [5,6]. In this case one should use as basic elements of expansion the Prym integrals $T_{\rho;Q_1Q_2} = -\int\limits_{Q_0}^{P} \tau_{\rho;Q_1^2Q_2^2}$ of second kind with two simple poles Q_1 and Q_2.

The Prym differential $\tau^{(2)}_{\rho;Q_1}$ admits the expansion $\left(\dfrac{1}{(z - z_1)^2} + \dfrac{c^{(1)}_{-1}}{z - z_1} + O(1) \right) dz$ in a neighborhood ofQ_1, $z(Q_1) = z_1$, where $c^{(1)}_{-1} = \sum\limits_{j=1}^{g} \log \rho(a_j) \varphi'_j(Q_1)$ [5,6].

The Prym differential $\tau^{(2)}_{\rho;Q_2}$ als has an expansion $\left(\dfrac{1}{(z - z_2)^2} + \dfrac{c^{(2)}_{-1}}{z - z_2} + O(1) \right) dz$ in a neighborhood of Q_2 на F_μ, where $c^{(2)}_{-1} = \sum\limits_{j=1}^{g} \log \rho(a_j) \varphi'_j(Q_2)$.

Prym differentials with two poles of the second order and zero residues at these points my be given in the form

$$\tau_{\rho;Q_1^2Q_2^2} = c^{(2)}_{-1} f_0(Q_1) \tau^{(2)}_{\rho;Q_1} - c^{(1)}_{-1} f_0(Q_2) \tau^{(2)}_{\rho;Q_2} - c^{(1)}_{-1} c^{(2)}_{-1} \tau_{\rho;Q_1Q_2}.$$

Note that the principal part for τ_{ρ,Q_1Q_2} at Q_1 has the form $\dfrac{f_0(Q_1)}{z-z_1}$, and at Q_2 it is $-\dfrac{f_0(Q_2)}{z-z_2}$. It follows that the differential constructed above $\tau_{\rho;Q_1^2Q_2^2}$ has poles of the second order at Q_1 and Q_2, and zero residues at these points.

Theorem 2. *Let f be a branch of a function of class M_1 for a non-essential character ρ on a variable Riemann surface $F_\mu^{'}$ of type $(g,n), g \geqslant 2,\ n > 0,$ with pairwise distinct poles at $P_{n+1},\dots,P_{n+s}$ of multiplicities $q_{n+1},\dots,\ q_{n+s}$ with given principal parts 1. Assume that for an analytic continuation of f to F_μ the conditions $(f) \geqslant \dfrac{1}{P_1^{q_1}\dots P_{n+s}^{q_{n+s}}}$, $q_j \geqslant 0, j = 1,\dots,n,$ and $\sum\limits_{j=1}^{g}\log\rho(a_j)\varphi_j^{'}(P_{n+s}) \neq 0$ are fulfilled. Then*

$$f(P) = \sum_{j=1}^{g} c_j \int_{Q_0}^{P} f_0\zeta_j + \sum_{r=1}^{n+s-1}\frac{A_{r1}T_{\rho;P_rP_{n+s}}}{d_{n+s}f_0(P_r)} + \sum_{m=2}^{q_1}\frac{A_{1m}}{(m-1)!}\frac{\partial^{m-1}T_{\rho;P_1P_{n+1}}}{\partial P_1^{m-1}} +$$

$$+\sum_{j=2}^{n+s}\left[A_{j,2}\frac{\partial T_{\rho;P_jP_1}}{\partial P_j} + \frac{A_{j,3}}{2!}\frac{\partial^2 T_{\rho;P_jP_1}}{\partial P_j^2} + \dots + \frac{A_{j,q_j}}{(q_j-1)!}\frac{\partial^{q_j-1}T_{\rho;P_jP_1}}{\partial P_j^{q_j-1}}\right] + C,$$

where

$$f = \frac{A_{j,q_j}}{(z-z(P_j))^{q_j}} + \dots + \frac{A_{j,2}}{(z-z(P_j))^2} + \frac{A_{j,1}}{z-z(P_j)} + O(1)$$

for some branch in a neighborhood of P_j, $j = 1,\dots,n+s$, on F_μ; $C = 0$ for $\rho \neq 1$; $d_k = \sum\limits_{m=1}^{g}\log\rho(a_m)\varphi_m^{'}(P_k)$, $k = 1,\dots,n+s$, on F_μ, and all summands depend holomorphically on $[\mu]$ and ρ.

3. Vector bundles of Prym differentials over a Techmüller space of Riemann surfaces of finite type

Denote by $\Omega_\rho^q\Big(\dfrac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}};F_\mu\Big)$ the vector space of (ρ,q)-differentials that are multiples of the divisor $\dfrac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}}$, where $\alpha_j \geqslant 1,\ \alpha_j \in \mathbb{N}, j = 1,\dots,s,\ s \geqslant 1, q \geqslant 1, q \in \mathbb{N}$, and by $\Omega_\rho^q(1;F_\mu)$ the vector subspace of holomorphic (ρ,q)-differentials on F_μ [2]. Here the divisor $Q_1\dots Q_s$ on F_μ is understood as a constant set of points on a surface F of genus $g \geqslant 2$.

Lemma 1 ([5], 105–106). *A holomorphic prinsipal $Hom(\Gamma,\mathbb{C}^*)$-bundle $\widetilde{E}$ is biholomorphic to the trivial bundle $\mathbb{T}_g(F_0)\times Hom(\Gamma,\mathbb{C}^*)$ over $\mathbb{T}_g(F_0)$.*

Proposition 1. *The vector bundle $E = \cup\Omega_\rho^q\Big(\dfrac{1}{Q_1^{\alpha_1}\cdots Q_s^{\alpha_s}};F_\mu\Big)/\Omega_\rho^q(1;F_\mu)$ over $\mathbb{T}_g\times(Hom(\Gamma,\mathbb{C}^*)\backslash 1)$ for $q > 1$ (over $\mathbb{T}_g\times(Hom(\Gamma,\mathbb{C}^*)\backslash L_g)$ when $q = 1$) and $g \geqslant 2$ is a holomorphic vector bundle of rank $\alpha_1+\dots+\alpha_s = d$, while the co-sets of (ρ,q)-differentials*

$$\tau_{\rho,q;Q_1}^{(1)},\dots,\tau_{\rho,q;Q_1}^{(\alpha_1)},\dots,\tau_{\rho,q;Q_s}^{(1)},\dots,\tau_{\rho,q;Q_s}^{(\alpha_s)}, \tag{2}$$

form a basis of locally holomorphic sections of this bundle.

Lemma 2. *For any divisor $P_1^{q_1}\cdot\ldots\cdot P_n^{q_n}$, $q_j \geqslant 0,\ j = 1,\dots,n,\ q > 1$ and any ρ (or $q = 1$ and an essential character ρ) on F_μ of genus $g \geqslant 2$, there exists a differential $\tilde{\omega} \in \Omega_\rho^q\Big(\dfrac{1}{P_1^{q_1}\cdot\ldots\cdot P_n^{q_n}},F_\mu\Big)$*

with the divisor $(\tilde{\omega}) = \dfrac{R_1, \dots, R_N}{P_1^{q_1} \cdot \ldots \cdot P_n^{q_n}}$, *where* $R_j \neq P_l, l = 1, \dots, n, j = 1, \dots, N$, $N = (2g-2)q +$ $+ q_1 + \ldots + q_n$, *and any given principal parts of Laurent series at* $P_j, j = 1, \dots, n$, *for its branches. This differential depends locally holomorphically on moduli* $[\mu]$ *of the surface* F_μ *and the character* ρ.

Consider the diagram

$$\begin{array}{ccc} E' = \cup \dfrac{\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_s^{\alpha_s}}, F'_\mu\right) \cap M_1}{\Omega^q_\rho(1, F'_\mu) \cap M_1} & \to & \cup \dfrac{\Omega^q_\rho\left(\frac{1}{Q_1^{\alpha_1} \ldots Q_s^{\alpha_s}}, F_\mu\right)}{\Omega^q_\rho(1, F_\mu)} = E \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{T}}^n_g \times Hom(\Gamma, \mathbb{C}^*) \backslash X & \to & \mathbb{T}_g \times Hom(\Gamma, \mathbb{C}^*) \backslash X, \end{array} \qquad (3)$$

where $\widetilde{\mathbb{T}}^n_g$ is a part of the Teichmüller space $\mathbb{T}_{g,n}$ [6, p. 81, p. 88], the vertical arrows are projections in vector bundles, and the lower horizontal arrow is related to the operation of gluing the punctures, which makes the surface $F \backslash \{P_1, \dots, P_n\}$ into a compact surface F [2].

Theorem 3. *The diagram above is a commutative diagram of vertical holomorphic vector bundles with isomorphic corresponding fibers and horizontal holomorphic* $n!$*-sheeted mappings, where* $X = 1$ *when* $q > 1$, *and* $X = L_g$ *when* $q = 1$.

It should be noted that analogous results hold true for the spaces of single-valued (Abelian) differentials.

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Дифференциалы Прима и пространства Тейхмюллера

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В работе исследуются мультипликативные мероморфные функции и дифференциалы на римановых поверхностях конечного типа. Доказан аналог формулы П. Аппеля о разложении мультипликативной функции с полюсами любых кратностей в сумму элементарных интегралов Прима. Построены явные базисы для ряда важных фактор-пространств. Доказана теорема о послойном изоморфизме векторных расслоений и $n!$*-листных отображений над пространствами Тейхмюллера. Эта теорема дает важную связь между пространствами дифференциалов Прима (абелевых дифференциалов) на компактной римановой поверхности и на римановой поверхности конечного типа.*

Ключевые слова: пространства Тейхмюллера римановых поверхностей конечного типа, дифференциалы Прима, векторные расслоения, группа характеров, многообразия Якоби.