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Singular Points of Complex Algebraic Hypersurfaces

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We consider a complex hypersurface V given by an algebraic equation in k unknowns, where the set $A \subset \mathbb{Z}^k$ of monomial exponents is fixed, and all the coefficients are variable. In other words, we consider a family of hypersurfaces in $(\mathbb{C} \setminus 0)^k$ parametrized by its coefficients $a = (a_\alpha)_{\alpha \in A} \in \mathbb{C}^A$. We prove that when A generates the lattice $\mathbb{Z}^k$ as a group, then over the set of regular points a in the A-discriminantal set, the singular points of V admit a rational expression in a.

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Introduction

By the *general algebraic hypersurface (or the A-hypersurface)* we mean the algebraic set V defined by the equation in k unknowns $y = (y_1, \ldots, y_k) \in (\mathbb{C} \setminus 0)^k$:

$$f(y_1, \ldots, y_k) := \sum_{\alpha=(\alpha_1,\ldots,\alpha_k)\in A} a_\alpha y_1^{\alpha_1} \ldots y_k^{\alpha_k} = 0. \tag{1}$$

Here $A \subset \mathbb{Z}^k$ is a fixed finite set while all coefficients a_α are treated as independent variables. We assume that the set A generates the lattice $\mathbb{Z}^k$ as a group. The set of polynomials (1) is identified with the space $\mathbb{C}^A$ of sequences $a = (a_\alpha)_{\alpha \in A}$ of dimension $N := \#A$. We can think about V as a family of hypersurfaces V_a in $(\mathbb{C} \setminus 0)^k$ parametrized by coefficients $a \in \mathbb{C}^A$.

The aim of the present paper is to obtain explicit formulas for almost all singular points of the hypersurface V. Recall that a point $y \in V$ is said to be singular if the polynomial f in (1) and all its partial derivatives $f'_{y_1}, \ldots, f'_{y_k}$ vanish at y. In the classical case, when $k = 1$, the following formulas were given in [1, Ch.1, Th.1.5]: if the equation

$$f(y) := a_d y^d + \ldots + a_1 y + a_0 = 0 \tag{2}$$

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has a unique multiple root $y = y(a) = y(a_0, \dots, a_d)$, and its multiplicity equals two, then y is given by rational expressions

$$y = \frac{\Delta_1(a)}{\Delta_0(a)} = \frac{\Delta_2(a)}{\Delta_1(a)} = \ldots = \frac{\Delta_d(a)}{\Delta_{d-1}(a)},$$

where $\Delta_j = \dfrac{\partial \Delta}{\partial a_j}$ are derivatives of the discriminant $\Delta(a)$ of the polynomial $f(y)$ in (2). Analogous formulas for a unique root of multiplicity $\nu \geqslant 2$ are given in [5], where instead of using the discriminant Δ, the resultant of f and its derivative $f^{(\nu-1)}$ (with respect to y) of order $\nu - 1$ is used.

We prove that almost all singular points $y(a)$ (actually, those that correspond to a belonging to the regular part of the discriminantal set) admit a rational representation (Theorem 3). In the last section we consider an example with comments how the type of a singular point $y(a) \in V$ depends of the singular type of $a \in \nabla_A$.

1. A-discriminant and the reduced equation

Definition 1 ([1]). *Let ∇° denote the set of all $(a_\alpha) \in \mathbb{C}^A$ such that the equation (1) has critical roots $y \in (\mathbb{C} \setminus 0)^k$, i.e. roots at which the gradient of f vanishes:*

$$f(y) = \frac{\partial f}{\partial y_1}(y) = \ldots = \frac{\partial f}{\partial y_k}(y) = 0.$$

The closure $\overline{\nabla^\circ} =: \nabla_A$ in $\mathbb{C}^A$ is said to be the A-discriminantal set.

In the set ∇_A is a hypersurface in $\mathbb{C}^A$, then by the *A-discriminant* one means an irreducible integral polynomial Δ_A in coefficients a of $f \in \mathbb{C}^A$ which vanishes on ∇_A.

The solution $y = y(a)$ to the equation (1) is $(k+1)$-homogeneous (it satisfies $k+1$ homogeneity conditions), and the A-discriminant inherits this property. To see this, we consider the following action on the space $\mathbb{C}^A$ of polynomials (1). For $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_k) \in (\mathbb{C} \setminus 0)^{k+1}$ we define it as follows

$$\lambda : \ f(y_1, \dots, y_k) \to \lambda_0 f(\lambda_1 y_1, \dots, \lambda_k y_k).$$

Observe that the set ∇_A is invariant under the λ–action. In terms of coefficients (a_α) of the polynomial f this action can be written in the following form:

$$a_\alpha \to \lambda_0 \lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k} a_\alpha, \quad \alpha \in A.$$

Here $\alpha_1, \dots, \alpha_k$ are the coordinates of α. In the toric part $(\mathbb{C} \setminus 0)^A \subset \mathbb{C}^A$ the orbits of this action are the equivalence classes with respect to the $(k+1)$-parametric subgroup defined by the immersion

$$(\lambda_0, \lambda_1, \dots, \lambda_k) \hookrightarrow \lambda_0 \lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k}, \quad \alpha \in A.$$

Its injectivity follows from the fact that A generates $\mathbb{Z}^k$. Renumerating the elements of A as $\alpha^1, \dots, \alpha^N$ we represent this immersion in the form

$$(a_\alpha) = \lambda^A,$$

where A is the matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_{11} & \alpha_{21} & \dots & \alpha_{N1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1k} & \alpha_{2k} & \dots & \alpha_{Nk} \end{pmatrix}, \tag{3}$$

and

$$\lambda^A = (\lambda^{a^1}, \dots, \lambda^{a^N}) = (\lambda_0 \lambda_1^{\alpha_{11}} \dots \lambda_k^{\alpha_{1k}}, \dots, \lambda_0 \lambda_1^{\alpha_{N1}} \dots \lambda_k^{\alpha_{Nk}})$$

with a^j being the columns of this matrix. Remark that we keep the notation A (which was used for the set of exponents α in (1)) for this extended matrix. Thus, an equivalence class can be written in the form $\lambda^A \cdot g$ with the coordinate-wise multiplication. In order to parameterize all equivalence classes we represent them in the form of an m–parametric subgroup

$$g = z^C, \quad z \in (\mathbb{C} \setminus 0)^m,$$

where C is an $m \times N$-matrix with $m = N - k - 1$. Choosing the matrix C in such a way that the $N \times N$-matrix

$$\widehat{A} = \left(\frac{A}{C}\right) \tag{4}$$

is unimodular (with determinant ± 1), we conclude that the transform

$$\widehat{A} : (\lambda, z) \to \lambda^A \cdot z^C$$

is an automorphism of the complex torus $(\mathbb{C} \setminus 0)^A$. Thus, for such C the m-parametric subgroup $g = z^C$ parametrizes all equivalence classes modulo the subgroup λ^A. Denoting by c^α the column of the matrix C indexed by an element $\alpha \in A$, we arrive at the following *reduced equation* for (1):

$$f(y) = \sum_{\alpha \in A} z^{c^\alpha} y^\alpha = 0, \tag{5}$$

where the coefficients $z^{c^\alpha} = z_1^{c_1^\alpha} \dots z_m^{c_m^\alpha}$, $\alpha \in A$, run over the m–parametric subgroup z^C in $(\mathbb{C} \setminus 0)^A$. The discriminantal set of the equation (5) we denote by ∇'_A and call it the *reduced discriminantal set*. The defining polynomial of ∇'_A is obtained from the A–discriminantal polynomial Δ_A. It is called the *reduced discriminant*.

By Kapranov's theorem [6] the reduced discriminantal set is birationally equivalent to the projective space $\mathbb{CP}^{m-1}$. Moreover, there is an explicit formula

$$z = (Bs)^B, \quad s \in \mathbb{CP}^{m-1}, \tag{6}$$

parametrizing ∇'_A. Clearly, then we get a parametrization of ∇_A as

$$a = (a_\alpha)_{\alpha \in A} = \lambda^A \cdot (Bs)^{BC}.$$

2. Parametrization of singular points

The matrix C, extending A in (4) defines a special matrix B, the so called Gale transform of A (see [1, P. 225]). Namely, the inverse of the matrix $\widehat{A}$ can be represented in the following block form:

$$\left(\widehat{A}\right)^{-1} = (D|B),$$

where D and B are blocks with $k+1$ and m columns, respectively. They satisfy the relations

$$A \cdot B = 0, \quad A \cdot D = E_{k+1}.$$

Remark that we can use columns α of A to index the rows for B writing them as b_α. With the help of B and D we can formulate the theorem on singular points of the reduced hypersurface (5).

The most convenient reductions of the equation (1) are associated with matrices C which contain $k+1$ zero columns at that the other m columns form the unit matrix. Such matrices can be used for extension of A to be unimodular if A has $k+1$ columns, say $\alpha^0, \alpha^1 \ldots, \alpha^k$, for which the columns

$$\alpha^1 - \alpha^0, \ldots, \alpha^k - \alpha^0$$

form a unimodular $k \times k$-matrix δ. In this case the reduction of (1) is just a fixation of the coefficients: $a_{\alpha^0} = a_{\alpha^1} = \ldots = a_{\alpha^k} = 1$. We can use such a reduction when δ is nondegenerate as well as in the case when δ is unimodular.

After dividing by y^{α^0} and denoting $\alpha^j - \alpha^0$ by α_j, $j = 1, \ldots, N-1$ we can assume that the reduction has the following form

$$f(y_1, \ldots, y_k) = 1 + \sum_{i=1}^{k} y_1^{\alpha_{i1}} \ldots y_k^{\alpha_{ik}} + \sum_{i=1}^{m} z_i y_1^{\alpha_{k+i,1}} \ldots y_k^{\alpha_{k+i,k}} = 0, \tag{7}$$

where the matrix $\delta = (\alpha_{ij}),\ i, j = 1, \ldots, k$ is nondegenerate. Let $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k$ be the first $k+1$ rows of the matrix B. In this case we have the following statement.

Theorem 1. *The vector-function $y(s) = (y_1(s), \ldots, y_k(s))$ with the coordinates*

$$y_j(s) = \prod_{\nu=1}^{k} \left(\frac{\langle \mathbf{b}_\nu, s \rangle}{\langle \mathbf{b}_0, s \rangle} \right)^{\chi_{j\nu}}, \quad j = 1, 2, \ldots, k,$$

where $\chi_{j\nu}$ are the entries of the matrix δ^{-1}, parameterizes the set of singular points of the reduced hypersurface (7).

Proof. Firstly, we consider the case when δ is the unit matrix, i.e. when f is of the type

$$f(y_1, \ldots, y_k) := 1 + y_1 + \ldots + y_k + \sum_{i=1}^{m} z_i y_1^{\alpha_{k+i,1}} \ldots y_k^{\alpha_{k+i,k}} = 0 \tag{8}$$

with an associated matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 0 & \alpha_{k+1,1} & \alpha_{k+2,1} & \ldots & \alpha_{N-1,1} \\ 0 & 0 & 1 & \ldots & 0 & \alpha_{k+1,2} & \alpha_{k+2,2} & \ldots & \alpha_{N-1,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & \alpha_{k+1,k} & \alpha_{k+2,k} & \ldots & \alpha_{N-1,k} \end{pmatrix}. \tag{9}$$

Choose the dual matrix

$$B = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \\ E_m \end{pmatrix},$$

where

$$b_0 = \Big(-1 + \sum_{j=1}^{k} \alpha_{k+1,j}; -1 + \sum_{j=1}^{k} \alpha_{k+2,j}; \ldots; -1 + \sum_{j=1}^{k} \alpha_{N-1,j} \Big)$$
$$b_1 = \Big(-\alpha_{k+1,1}; -\alpha_{k+2,1}; \ldots; -\alpha_{N-1,1} \Big)$$
$$\ldots$$
$$b_k = \Big(-\alpha_{k+1,k}; -\alpha_{k+2,k}; \ldots; -\alpha_{N-1,k} \Big).$$

Due to the Horn-Kapranov formula (6) the discriminantal set of the equation (8) is given by the following parametrization

$$z_i = s_i \langle b_0, s\rangle^{-1+\sum\limits_{j=1}^{k} \alpha_{k+i,j}} \langle b_1, s\rangle^{-\alpha_{k+i,1}} \ldots \langle b_k, s\rangle^{-\alpha_{k+i,k}},\ i = 1, 2 \ldots, m,$$

where $s = (s_1, \ldots, s_m)$.

Lemma 1. *The vector-function $y(s) = (y_1(s), \ldots y_k(s))$ with coordinates*

$$y_1(s) = \frac{\langle b_1, s\rangle}{\langle b_0, s\rangle}, \ldots, y_k(s) = \frac{\langle b_k, s\rangle}{\langle b_0, s\rangle}$$

satisfies the system of equations

$$f(y_1, \ldots, y_k) = \frac{\partial f(y_1, \ldots, y_k)}{\partial y_1} = \ldots = \frac{\partial f(y_1, \ldots, y_k)}{\partial y_k} = 0. \tag{10}$$

Proof. Let us substitute $y = y(s)$ into the equation (8) with the coefficients $z = = (Bs)^B$. We get the following expression

$$1 + \frac{\langle b_1, s\rangle}{\langle b_0, s\rangle} + \ldots + \frac{\langle b_k, s\rangle}{\langle b_0, s\rangle} +$$
$$+ \sum_{i=1}^{m} \left(s_i \langle b_0, s\rangle^{-1+\sum\limits_{j=1}^{k} \alpha_{k+i,j}} \langle b_1, s\rangle^{-\alpha_{k+i,1}} \ldots \langle b_k, s\rangle^{-\alpha_{k+i,k}} \right) \left(\frac{\langle b_1, s\rangle}{\langle b_0, s\rangle} \right)^{\alpha_{k+i,1}} \ldots \left(\frac{\langle b_k, s\rangle}{\langle b_0, s\rangle} \right)^{\alpha_{k+i,k}} =$$
$$= \frac{\langle b_0, s\rangle + \langle b_1, s\rangle + \ldots + \langle b_k, s\rangle}{\langle b_0, s\rangle} + \frac{s_1 + \ldots + s_m}{\langle b_0, s\rangle}.$$

The last sum vanishes, since

$$\langle b_0, s\rangle + \langle b_1, s\rangle + \ldots + \langle b_k, s\rangle = -(s_1 + \ldots + s_m).$$

Recall that the sum of all rows of the matrix B is equal to zero, and B consists of rows $b_0, \ldots, b_k$ supplemented by the unit $m \times m$–matrix. So, $y(s)$ annihilates $f(y)$ when $z = (Bs)^B$.

Similarly for the derivatives, one has as follows

$$\frac{\partial g(y_1(s), \ldots y_k(s))}{\partial y_j} = 1 + \frac{1}{\langle b_0, s\rangle} \sum_{i=1}^{m} \alpha_{k+i,j} s_i = \frac{1}{\langle b_j, s\rangle} \Big(\langle b_j, s\rangle + \sum_{i=1}^{m} \alpha_{k+i,j} s_i \Big) = 0.$$

The last expression vanishes due to the property of vectors b_j. So, the proof of Lemma 1 is completed. □

In order to continue the proof of Theorem 1 let us turn to the equation (7). We introduce the monomial change

$$x_i = y_1^{\alpha_{i1}} \ldots y_k^{\alpha_{ik}},\ i = 1, 2, \ldots, k,$$

which can be rewritten in the matrix form as $x = y^\delta$. Since δ is nondegenerate, one has

$$y = x^{\delta^{-1}}.$$

Let us write the matrix $A = (\alpha_{ij})$ in the block form $A = (\delta, \delta')$. Then after the substitution $y = x^{\delta^{-1}}$ in (7) we get

$$1 + \sum_{i=1}^{k} x_i + \sum_{i=1}^{m} z_i (x^{\delta^{-1}\delta'})_i = 0, \tag{11}$$

where $(x^{\delta^{-1}\delta'})_i$ is the i-th coordinate of the vector $x^{\delta^{-1}\delta'}$. The exponents in equation (11) form the $k \times N$-matrix $(E_k, \delta^{-1}\delta')$. This matrix supplemented by the row of units looks like (9) where the block δ' is changed by $\delta^{-1}\delta'$:

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & \dots & 1 & | & 1 & \dots & 1 \\ 0 & & & & | & & & \\ \vdots & & E_k & & | & & \delta^{-1}\delta' & \\ 0 & & & & | & & & \end{pmatrix}.$$

The computation shows that the dual matrix to A is the matrix

$$\mathcal{B} = \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_k \\ E_m \end{pmatrix}.$$

Further applying Lemma 1 we complete the proof of Theorem 1. □

3. Rational expression for singular points

As it follows from the definition of the A-discriminantal set, the singular points of the hypersurface which we consider coincide with the restrictions of solutions to the equation (1) on the A-discriminantal set, i.e. with $y(a)\big|_{\nabla_A}$. For the reduced equation (7) the singular points $y(z)$ are given by

$$y(z(s)) = y((Bs)^B).$$

However, according to Kapranov's theorem [6] the parametrization $z = (Bs)^B$ is the inverse of the logarithmic Gauss map

$$\gamma : \ \nabla'_A \to \mathbb{CP}^{m-1}$$

of a reduced A-discriminantal set ∇'_A. At the regular points $z \in reg\nabla'_A$ this mapping can be written explicitly (see [8])

$$\gamma : \ z \to (z_1(\Delta')_{z_1} : \ldots : z_m(\Delta')_{z_m}) = (s_1 : \ldots : s_m),$$

where for simplicity we write Δ' instead of Δ'_A. Therefore, by Theorem 1 we get the following statement.

Theorem 2. *The singular points of the reduced hypersurface (7) over the set* $reg\,\nabla'_A$ *admit in global coordinates* z *the following radical representation:*

$$y_j(z) = \prod_{\nu=1}^{k} \left(\frac{\langle b_\nu, \gamma(z)\rangle}{\langle b_0, \gamma(z)\rangle} \right)^{\chi_{j\nu}}, \quad j = 1, 2, \ldots, k, \tag{12}$$

where $\chi_{j\nu}$ *are the entries of the matrix* δ^{-1}.

Now we can formulate the main result.

Theorem 3. *Let the set* A *in (1) generate* $\mathbb{Z}^k$ *as a group. The singular points* $y(a)$ *of the hypersurface (1) over the set* $\operatorname{reg}\nabla_A$ *admit a rational representation.*

Proof. We consider an arbitrary reduction of the type (7) with fixed coefficients $a_{\alpha^{j_0}} = \ldots = a_{\alpha^{j_k}} = 1$. Let $B_{J'}$ be the submatrix of the dual matrix B consisting of rows $b_{\alpha^{j_1}}, \ldots, b_{\alpha^{j_k}}$. Then by Theorem 2 the singular points of the reduced hypersurface can be found in the following way:

$$y(z) = \left(\frac{\langle B_{J'}, \gamma(z)\rangle}{\langle b_{j_0}, \gamma(z)\rangle}\right)^{\alpha_J^{-1}}.$$

Consider all subsets $J = \{j_0, j_1, \ldots, j_k\} \subset \{1, \ldots, N\}$ for which the corresponding matrices δ_J are nondegenerate. Then there exist integer numbers q_J such that

$$\sum_J q_J \delta_J = E_k.$$

Consequently, we have

$$y(a) = y^{E_k} = y^{\sum\limits_J q_J\delta_J} = \prod_J \left(\frac{\langle B_{J'}, \gamma(z)\rangle}{\langle b_{j_0}, \gamma(z)\rangle}\right)^{\delta_J^{-1} q_J \delta_J} = \prod_J \left(\frac{\langle B_{J'}, \gamma(z)\rangle}{\langle b_{j_0}, \gamma(z)\rangle}\right)^{q_J}.$$

The last term is a rational expression in variables z. Since by Kapranov's theorem γ is a birational map we get rationality of $y(a)$ in variables a. □

4. Example

Let us consider the following polynomial equation

$$a_{00} + a_{10}y_1 + a_{01}y_2 + a_{31}y_1^3y_2 + a_{63}y_1^6y_2^3 = 0.$$

It is associated with the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix},$$

which has the right annulator

$$B = \begin{pmatrix} 3 & 8 \\ -3 & -6 \\ -1 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The reduced equation looks as follows:

$$f = 1 + y_1 + y_2 + z_1y_1^3y_2 + z_2y_1^6y_2^3 = 0. \tag{13}$$

According to (6) the parametrization of the reduced A-discriminantal set $\nabla' = \nabla'_A$ for f is

$$z_1 = s_1(3s_1+8s_2)^3(-3s_1-6s_2)^{-3}(-s_1-3s_2)^{-1} = \frac{(3+8s)^3}{(3+6s)^3(1+3s)}, \tag{14}$$

$$z_2 = s_2(3s_1+8s_2)^8(-3s_1-6s_2)^{-6}(-s_1-3s_2)^{-3} = -\frac{s(3+8s)^8}{(3+6s)^6(1+3s)^3}, \tag{15}$$

where $s := \dfrac{s_2}{s_1}$ is an affine coordinate in $\mathbb{CP}_1$. After elimination of the parameter s in the system (14)–(15) we get the reduced A-discriminant $\Delta' = \Delta'_A$:

$$\Delta' = -262144z_2^3 + 331776z_1z_2^3 + 331776z_1^3z_2^2 - 61236z_1^6z_2 - 61236z_1^2z_2^3 - 19683z_2^4 - 398034z_1^4z_2^2 +$$
$$+59049z_1^7z_2 + 19683z_1^3z_2^3 + 59049z_1^5z_2^2 - 19683z_1^8 + 19683z_1^9.$$

The matrix δ for the equation (12) is the unit matrix, therefore by (12) we get the following formulas for singular points:

$$y_1 = \frac{-3z_1(\Delta')_{z_1} - 6z_2(\Delta')_{z_2}}{3z_1(\Delta')_{z_1} + 8z_2(\Delta')_{z_2}}, \quad y_2 = \frac{-z_1(\Delta')_{z_1} - 3z_2(\Delta')_{z_2}}{3z_1(\Delta')_{z_1} + 8z_2(\Delta')_{z_2}}.$$

The derivatives

$$\frac{dz_1}{ds} = -\frac{(3+8s)^2(4s+1)^2}{9(1+2s)^4(1+3s)^2}, \quad \frac{dz_2}{ds} = -\frac{(3+8s)^7(4s+1)^2}{243(1+2s)^7(1+3s)^4}$$

vanish when $s = -\dfrac{1}{4}$ and $s = -\dfrac{3}{8}$. It means that ∇_A has two singular points

$$z\Big(-\frac{1}{4}\Big) = \Big(\frac{32}{27}, \frac{1024}{729}\Big) \ \ and \ \ z\Big(-\frac{3}{8}\Big) = (0,0).$$

Elimination of the parameter s in the system (14)–(15) leads us to the A-discriminant $\Delta' = \Delta'_A$:

$$\Delta' = -262144z_2^3 + 331776z_1z_2^3 + 331776z_1^3z_2^2 - 61236z_1^6z_2 - 61236z_1^2z_2^3 - 19683z_2^4 - 398034z_1^4z_2^2 +$$
$$+59049z_1^7z_2 + 19683z_1^3z_2^3 + 59049z_1^5z_2^2 - 19683z_1^8 + 19683z_1^9.$$

Consider the Taylor decomposition of Δ' at the point $z(-\frac{1}{4})$, i.e. by powers of $p = z_1 - \dfrac{32}{27}$ and $q = z_2 - \dfrac{1024}{729}$:

$$\Delta' = \frac{68719476736}{19683}p^3 - \frac{536870912}{243}p^2q + \frac{4194304}{9}pq^2 - 32768q^3 + \frac{486539264}{81}p^4 + r(p,q),$$

where $r(p,q)$ is a sum of monomials of degree $\geqslant 4$ except the monomial p^4. Here the initial homogeneous cubic form is a cube power of an affine polynomial

$$\frac{32768}{14348907}(1152z_1 - 243z_2 - 1024)^3.$$

Consequently, in coordinates $m = 1152z_1 - 243z_2 - 1024$ and $l = z_1 - \dfrac{32}{27}$ the discriminant has the form

$$\Delta' = am^3 + bl^4 + \ldots, \quad a \neq 0, \ b \neq 0,$$

It means that $z = (32/27, 1024/729)$ is a cuspidal point of the type $(4, 3)$ for the discriminant Δ'.

Now we have to study singular types of singular points of the complex curve (13) which are given by Theorem 2:

$$y_1(s) = \frac{-3-6s}{3+8s}, \quad y_2(s) = \frac{-1-3s}{3+8s}.$$

At the singular points $y(s)$ we have the following expression for the Hessian of f:

$$\frac{\partial^2 f}{\partial y_1^2}\frac{\partial^2 f}{\partial y_2^2} - \Big(\frac{\partial^2 f}{\partial y_1 \partial y_2}\Big)^2 = -\frac{(3+8s)^2(1+4s)^2}{(1+2s)^2(1+3s)^2}.$$

Therefore, only $y(-\frac{1}{4}) = (-\frac{3}{2}, -\frac{1}{4})$ is not a Morse point.

Consider the expression of the polynomial (13) at the point $y(-\frac{1}{4})$:

$$f = -12(y_2+1/4)^2 - 4(y_1+3/2)(y_2+1/4) - \frac{1}{3}(y_1+3/2)^2 + 16(y_2+1/4)^3 +$$

$$+48(y_1+3/2)(y_2+1/4)^2 + \frac{44}{3}(y_1+3/2)^2(y_2+1/4) + 32/27(y_1+3/2)^3 - \frac{448}{27}(y_1+3/2)^3(y_2+1/4) -$$

$$-64(y_1+3/2)(y_2+1/4)^3 - 80(y_1+3/2)^2(y_2+1/4)^2 - \frac{20}{27}(y_1+3/2)^4 + \frac{320}{3}(y_1+3/2)^2(y_2+1/4)^3 +$$

$$+\frac{640}{9}(y_1+3/2)^3(y_2+1/4)^2 + \frac{16}{81}(y_1+3/2)^5 + \frac{80}{9}(y_1+3/2)^4(y_2+1/4) + \ldots.$$

After the change of variables

$$y_1 + \frac{3}{2} = -\frac{u}{15} + \frac{v}{30}, \quad y_2 + \frac{1}{4} = \frac{8u}{45} - \frac{v}{180}$$

we get

$$f = \frac{1}{3}u^2 + \frac{1}{8201250}v^4 + r(u, v),$$

where $r(u, v)$ consists of monomials of weighted degree $\geqslant 4$ with respect to the weight $(2, 1)$. This means that $y(-\frac{1}{4})$ is a self-intersection point for the curve (13) with a common tangent line.

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Сингулярные точки комплексных алгебраических гиперповерхностей

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Рассматривается комплексная гиперповерхность V, заданная алгебраическим уравнением с k неизвестными и с переменными коэффициентами, причем множество $A \subset \mathbb{Z}^k$ показателей мономов уравнения произвольное, но фиксированное. Таким образом, мы рассматриваем семейство гиперповерхностей, параметризованных наборами коэффициентов $a = (a_\alpha)_{\alpha \in A} \in \mathbb{C}^A$. Доказывается, что если A порождает решетку $\mathbb{Z}^k$ как группу, то над множеством регулярных точек A-дискриминантного множества сингулярные точки гиперповерхности V рационально выражаются через коэффициенты a.

Ключевые слова: особая точка, A-дискриминант, логарифмическое отображение Гаусса.