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Hall's Polynomials of Finite Two-Generator Groups of Exponent Seven

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Let $B_k = B_0(2,7,k)$ be the largest two-generator finite group of exponent 7 and nilpotency class k. Hall's polynomials of B_k for $k \leqslant 4$ are calculated.

Keywords: periodic group, collection process, Hall's polynomials.

Let $B_k = B_0(2,7,k)$ be the largest two-generator finite group of exponent 7 and nilpotency class k. In this class, the largest group is the group B_{28}, which has the order 7^{20416} [1]. For each B_k a power commutator presentation is obtained [1].

Let $a_1^{x_1} \dots a_n^{x_n}$ and $a_1^{y_1} \dots a_n^{y_n}$ be two arbitrary elements in the group B_k recorded in the commutator form. Then their product is equal

$$a_1^{x_1} \dots a_n^{x_n} \cdot a_1^{y_1} \dots a_n^{y_n} = a_1^{z_1} \dots a_n^{z_n}.$$

Powers z_i are to be found based on the collection process (see [2,3]) which is implemented in the computer algebra systems GAP and MAGMA. Furthermore, there is an alternative method for calculating products of elements of the group, proposed by Hall (see [4]). Hall showed that z_i are polynomial functions (over the field $\mathbb{Z}_7$ in this case), depending on the variables $x_1, \dots, x_i, y_1, \dots, y_i$, which is now called Hall's polynomials. According to [4]

$$z_i = x_i + y_i + p_i(x_1, \dots, x_{i-1}, y_1, \dots, y_{i-1}).$$

Hall's polynomials are necessary in solving problems that require multiple products of the elements of the group. Study of the structure of the Cayley graph of some group is one of these problems [5,6]. The computational experiments carried out on the computer in two-generator groups of exponent five (see [7]) showed that the method of Hall's polynomials has an advantage over the traditional collection process. Therefore, there is a reason to believe that the use of polynomials would be preferable than the collection process in the study of Cayley graphs of B_k groups. It should also be noted that this method is easily software-implemented including multiprocessor computer systems.

Previously unknown Hall's polynomials of B_k are calculated within the framework of this paper. For $k > 4$ polynomials are calculated similarly but their output takes considerably more space so it makes impossible to verify the proof without use of computers.

The main result of this paper is

Theorem. *Let $a_1^{x_1} \dots a_n^{x_n}$ and $a_1^{y_1} \dots a_n^{y_n}$ be two arbitrary elements of the group B_k recorded in the commutator form where $k \in \mathbb{N}$ u $k \leqslant 4$. Then their product is equal $a_1^{x_1} \dots a_n^{x_n} \cdot a_1^{y_1} \dots a_n^{y_n} =$*

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$a_1^{z_1} \dots a_n^{z_n}$, where $z_i \in \mathbb{Z}_7$, are Hall's polynomials given by formulas (1–2) for $k = 1$, (1–3) for $k = 2$, (1–5) for $k = 3$ and (1–8) for $k = 4$.

$$z_1 = x_1 + y_1, \tag{1}$$

$$z_2 = x_2 + y_2, \tag{2}$$

$$z_3 = x_3 + y_3 + x_2y_1, \tag{3}$$

$$z_4 = x_4 + y_4 + 3x_2y_1 + x_3y_1 + 4x_2y_1^2, \tag{4}$$

$$z_5 = x_5 + y_5 + 3x_2y_1 + x_3y_2 + 4x_2^2y_1 + x_2y_1y_2, \tag{5}$$

$$z_6 = x_6 + y_6 + 5x_2y_1 + 3x_3y_1 + x_4y_1 + 3x_2y_1^2 + 6x_2y_1^3 + 4x_3y_1^2, \tag{6}$$

$$z_7 = x_7 + y_7 + 2x_2^2y_1^2 + 2x_2y_1 + x_4y_2 + x_5y_1 + 5x_2y_1^2 + 5x_2^2y_1 + 4x_2y_1^2y_2 + 3x_2y_1y_2 + x_3y_1y_2, \tag{7}$$

$$z_8 = x_8 + y_8 + 5x_2y_1 + 3x_3y_2 + x_5y_2 + 3x_2^2y_1 + 6x_2^3y_1 + 4x_3y_2^2 + 4x_2y_1y_2^2 + 4x_2^2y_1y_2 + 6x_2y_1y_2. \tag{8}$$

1. Proof of the Theorem

Let's calculate the Hall's polynomials for the group B_4. Dealing with this group we also obtain polynomials for groups B_1, B_2 and B_3 so no need to study separately these cases. When $k < 4$ commutators which has a weight is more than k are not considered because of they are definitionally equal to the group identity.

Using GAP we obtain a power commutator presentation of B_4.

Commutators of weight 1:

a_1, a_2 — generators of the group.

Commutators of weight 2:

$$a_3 = [a_2, a_1].$$

Commutators of weight 3:

$$a_4 = [a_3, a_1] = [a_2, a_1, a_1],$$

$$a_5 = [a_3, a_2] = [a_2, a_1, a_2].$$

Commutators of weight 4:

$$a_6 = [a_4, a_1] = [a_2, a_1, a_1, a_1],$$

$$a_7 = [a_5, a_1] = [a_2, a_1, a_2, a_1],$$

$$a_8 = [a_5, a_2] = [a_2, a_1, a_2, a_2].$$

List of defining relations R for commutators:

$a_i^7 = 1$ $(1 \leqslant i \leqslant 8)$, $[a_2, a_1] = a_3$, $[a_3, a_1] = a_4$, $[a_3, a_2] = a_5$, $[a_4, a_1] = a_6$,

$[a_4, a_2] = a_7$, $[a_4, a_3] = 1$, $[a_5, a_1] = a_7$, $[a_5, a_2] = a_8$, $[a_5, a_3] = 1$, $[a_5, a_4] = 1$,

$[a_6, a_1] = 1$, $[a_6, a_2] = 1$, $[a_6, a_3] = 1$, $[a_6, a_4] = 1$, $[a_6, a_5] = 1$, $[a_7, a_1] = 1$,

$[a_7, a_2] = 1$, $[a_7, a_3] = 1$, $[a_7, a_4] = 1$, $[a_7, a_5] = 1$, $[a_7, a_6] = 1$, $[a_8, a_1] = 1$,

$[a_8, a_2] = 1$, $[a_8, a_3] = 1$, $[a_8, a_4] = 1$, $[a_8, a_5] = 1$, $[a_8, a_6] = 1$, $[a_8, a_7] = 1$.

Thus,

$$B_4 = \langle\, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \mid R \,\rangle.$$

Each element of the group is expressed uniquely as a normal commutator word:

$$\forall g \in B_4 \quad g = a_1^{x_1} a_2^{x_2} a_3^{x_3} a_4^{x_4} a_5^{x_5} a_6^{x_6} a_7^{x_7} a_8^{x_8}, \quad x_i \in \mathbb{Z}_7.$$

Sometimes we will write $g = (x_1, \ldots, x_8)$.

In order to determine the functions z_i first we need to calculate the products of $a_j^y\, a_i^x$ for all $1 \leqslant i < j \leqslant 8$, $x, y = 1, 2, 3, 4, 5, 6$. For the pair (j, i) it is required to find the interpolation polynomial for each of the 8 commutators by the 36 values of the product (y, x).

Let's start with the first pair $a_2^y\, a_1^x$:

$$a_2^1 a_1^1 = (1,1,1,0,0,0,0,0), \quad a_2^1 a_1^2 = (2,1,2,1,0,0,0,0), \quad a_2^1 a_1^3 = (3,1,3,3,0,1,0,0),$$

$$a_2^1 a_1^4 = (4,1,4,6,0,4,0,0), \quad a_2^1 a_1^5 = (5,1,5,3,0,3,0,0), \quad a_2^1 a_1^6 = (6,1,6,1,0,6,0,0),$$

$$a_2^2 a_1^1 = (1,2,2,0,1,0,0,0), \quad a_2^2 a_1^2 = (2,2,4,2,2,0,1,0), \quad a_2^2 a_1^3 = (3,2,6,6,3,2,3,0),$$

$$a_2^2 a_1^4 = (4,2,1,5,4,1,6,0), \quad a_2^2 a_1^5 = (5,2,3,6,5,6,3,0), \quad a_2^2 a_1^6 = (6,2,5,2,6,5,1,0),$$

$$a_2^3 a_1^1 = (1,3,3,0,3,0,0,1), \quad a_2^3 a_1^2 = (2,3,6,3,6,0,3,2), \quad a_2^3 a_1^3 = (3,3,2,2,2,3,2,3),$$

$$a_2^3 a_1^4 = (4,3,5,4,5,5,4,4), \quad a_2^3 a_1^5 = (5,3,1,2,1,2,2,5), \quad a_2^3 a_1^6 = (6,3,4,3,4,4,3,6),$$

$$a_2^4 a_1^1 = (1,4,4,0,6,0,0,4), \quad a_2^4 a_1^2 = (2,4,1,4,5,0,6,1), \quad a_2^4 a_1^3 = (3,4,5,5,4,4,4,5),$$

$$a_2^4 a_1^4 = (4,4,2,3,3,2,1,2), \quad a_2^4 a_1^5 = (5,4,6,5,2,5,4,6), \quad a_2^4 a_1^6 = (6,4,3,4,1,3,6,3),$$

$$a_2^5 a_1^1 = (1,5,5,0,3,0,0,3), \quad a_2^5 a_1^2 = (2,5,3,5,6,0,3,6), \quad a_2^5 a_1^3 = (3,5,1,1,2,5,2,2),$$

$$a_2^5 a_1^4 = (4,5,6,2,5,6,4,5), \quad a_2^5 a_1^5 = (5,5,4,1,1,1,2,1), \quad a_2^5 a_1^6 = (6,5,2,5,4,2,3,4),$$

$$a_2^6 a_1^1 = (1,6,6,0,1,0,0,6), \quad a_2^6 a_1^2 = (2,6,5,6,2,0,1,5), \quad a_2^6 a_1^3 = (3,6,4,4,3,6,3,4),$$

$$a_2^6 a_1^4 = (4,6,3,1,4,3,6,3), \quad a_2^6 a_1^5 = (5,6,2,4,5,4,3,2), \quad a_2^6 a_1^6 = (6,6,1,6,6,1,1,1).$$

Let's write:

$$a_2^y\, a_1^x = a_1^x\, a_2^y\, a_3^{f_3^{(1,2)}(x,y)}\, a_4^{f_4^{(1,2)}(x,y)} \ldots a_8^{f_8^{(1,2)}(x,y)},$$

where $f_r^{(1,2)}(x,y) = \sum_{p=1}^{6} \sum_{q=1}^{6} \beta_{pq}^r x^p y^q$ are some polynomials over the field $\mathbb{Z}_7$. To find them let's perform interpolation for each commutator $r = 3, 4, \ldots, 8$.

To find $f_r^{(1,2)}(x,y)$ it is required to solve a system of linear equations over the given field:

$$\sum_{p=1}^{6} \sum_{q=1}^{6} \beta_{pq}^r x^p y^q = z_r^{yx} \quad \forall\, x, y = 1, 2, 3, 4, 5, 6, \tag{9}$$

where z_r^{yx} is a value of r-th commutator for the pair (y, x). This system will have 36 variables and consist of 36 equations.

Let's show how to find $f_8^{(1,2)}(x,y)$ at the example of the 8-th commutator. For short, let's write β_{pq} instead of β_{pq}^8. Substituting in (9) all values of z_8^{yx} we receive:

$$\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
2&2&2&2&2&2&4&4&4&4&4&4&1&1&1&1&1&1&2&2&2&2&2&2&4&4&4&4&4&4&1&1&1&1&1&1\\
3&3&3&3&3&3&2&2&2&2&2&2&6&6&6&6&6&6&4&4&4&4&4&4&5&5&5&5&5&5&1&1&1&1&1&1\\
4&4&4&4&4&4&2&2&2&2&2&2&1&1&1&1&1&1&4&4&4&4&4&4&2&2&2&2&2&2&1&1&1&1&1&1\\
5&5&5&5&5&5&4&4&4&4&4&4&6&6&6&6&6&6&2&2&2&2&2&2&3&3&3&3&3&3&1&1&1&1&1&1\\
6&6&6&6&6&6&1&1&1&1&1&1&6&6&6&6&6&6&1&1&1&1&1&1&6&6&6&6&6&6&1&1&1&1&1&1\\
2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1&2&4&1\\
4&1&2&4&1&2&1&2&4&1&2&4&2&4&1&2&4&1&4&1&2&4&1&2&1&2&4&1&2&4&2&4&1&2&4&1\\
6&5&3&6&5&3&4&1&2&4&1&2&5&3&6&5&3&6&1&2&4&1&2&4&3&6&5&3&6&5&2&4&1&2&4&1\\
1&2&4&1&2&4&4&1&2&4&1&2&2&4&1&2&4&1&1&2&4&1&2&4&4&1&2&4&1&2&2&4&1&2&4&1\\
3&6&5&3&6&5&1&2&4&1&2&4&5&3&6&5&3&6&4&1&2&4&1&2&6&5&3&6&5&3&2&4&1&2&4&1\\
5&3&6&5&3&6&2&4&1&2&4&1&5&3&6&5&3&6&2&4&1&2&4&1&5&3&6&5&3&6&2&4&1&2&4&1\\
3&2&6&4&5&1&3&2&6&4&5&1&3&2&6&4&5&1&3&2&6&4&5&1&3&2&6&4&5&1&3&2&6&4&5&1\\
6&4&5&1&3&2&5&1&3&2&6&4&3&2&6&4&5&1&6&4&5&1&3&2&5&1&3&2&6&4&3&2&6&4&5&1\\
2&6&4&5&1&3&6&4&5&1&3&2&4&5&1&3&2&6&5&1&3&2&6&4&1&3&2&6&4&5&3&2&6&4&5&1\\
5&1&3&2&6&4&6&4&5&1&3&2&3&2&6&4&5&1&5&1&3&2&6&4&6&4&5&1&3&2&3&2&6&4&5&1\\
1&3&2&6&4&5&5&1&3&2&6&4&4&5&1&3&2&6&6&4&5&1&3&2&2&6&4&5&1&3&3&2&6&4&5&1\\
4&5&1&3&2&6&3&2&6&4&5&1&4&5&1&3&2&6&3&2&6&4&5&1&4&5&1&3&2&6&3&2&6&4&5&1\\
4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1&4&2&1\\
1&4&2&1&4&2&2&1&4&2&1&4&4&2&1&4&2&1&1&4&2&1&4&2&2&1&4&2&1&4&4&2&1&4&2&1\\
5&6&3&5&6&3&1&4&2&1&4&2&3&5&6&3&5&6&2&1&4&2&1&4&6&3&5&6&3&5&4&2&1&4&2&1\\
2&1&4&2&1&4&1&4&2&1&4&2&4&2&1&4&2&1&2&1&4&2&1&4&1&4&2&1&4&2&4&2&1&4&2&1\\
6&3&5&6&3&5&2&1&4&2&1&4&3&5&6&3&5&6&1&4&2&1&4&2&5&6&3&5&6&3&4&2&1&4&2&1\\
3&5&6&3&5&6&4&2&1&4&2&1&3&5&6&3&5&6&4&2&1&4&2&1&3&5&6&3&5&6&4&2&1&4&2&1\\
5&4&6&2&3&1&5&4&6&2&3&1&5&4&6&2&3&1&5&4&6&2&3&1&5&4&6&2&3&1&5&4&6&2&3&1\\
3&1&5&4&6&2&6&2&3&1&5&4&5&4&6&2&3&1&3&1&5&4&6&2&6&2&3&1&5&4&5&4&6&2&3&1\\
1&5&4&6&2&3&3&1&5&4&6&2&2&3&1&5&4&6&6&2&3&1&5&4&4&6&2&3&1&5&5&4&6&2&3&1\\
6&2&3&1&5&4&3&1&5&4&6&2&5&4&6&2&3&1&6&2&3&1&5&4&3&1&5&4&6&2&5&4&6&2&3&1\\
4&6&2&3&1&5&6&2&3&1&5&4&2&3&1&5&4&6&3&1&5&4&6&2&1&5&4&6&2&3&5&4&6&2&3&1\\
2&3&1&5&4&6&5&4&6&2&3&1&2&3&1&5&4&6&5&4&6&2&3&1&2&3&1&5&4&6&5&4&6&2&3&1\\
6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1&6&1\\
5&2&5&2&5&2&3&4&3&4&3&4&6&1&6&1&6&1&5&2&5&2&5&2&3&4&3&4&3&4&6&1&6&1&6&1\\
4&3&4&3&4&3&5&2&5&2&5&2&1&6&1&6&1&6&3&4&3&4&3&4&2&5&2&5&2&5&6&1&6&1&6&1\\
3&4&3&4&3&4&5&2&5&2&5&2&6&1&6&1&6&1&3&4&3&4&3&4&5&2&5&2&5&2&6&1&6&1&6&1\\
2&5&2&5&2&5&3&4&3&4&3&4&1&6&1&6&1&6&5&2&5&2&5&2&4&3&4&3&4&3&6&1&6&1&6&1\\
1&6&1&6&1&6&6&1&6&1&6&1&1&6&1&6&1&6&6&1&6&1&6&1&1&6&1&6&1&6&6&1&6&1&6&1
\end{bmatrix}
\cdot
\begin{pmatrix}
\beta_{11}\\ \beta_{12}\\ \beta_{13}\\ \beta_{14}\\ \beta_{15}\\ \beta_{16}\\ \beta_{21}\\ \beta_{22}\\ \beta_{23}\\ \beta_{24}\\ \beta_{25}\\ \beta_{26}\\ \beta_{31}\\ \beta_{32}\\ \beta_{33}\\ \beta_{34}\\ \beta_{35}\\ \beta_{36}\\ \beta_{41}\\ \beta_{42}\\ \beta_{43}\\ \beta_{44}\\ \beta_{45}\\ \beta_{46}\\ \beta_{51}\\ \beta_{52}\\ \beta_{53}\\ \beta_{54}\\ \beta_{55}\\ \beta_{56}\\ \beta_{61}\\ \beta_{62}\\ \beta_{63}\\ \beta_{64}\\ \beta_{65}\\ \beta_{66}
\end{pmatrix}
=
\begin{pmatrix}
0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\2\\3\\4\\5\\6\\4\\1\\5\\2\\6\\3\\3\\6\\2\\5\\1\\4\\6\\5\\4\\3\\2\\1
\end{pmatrix}.$$

The rank of the system matrix is equal to 36, therefore it has the only solution: $\beta_{11} = 5$, $\beta_{12} = 3$, $\beta_{13} = 6$, all the remaining coefficients are equal to zero. Therefore,

$$f_8^{(1,2)}(x,y) = 5xy + 3xy^2 + 6xy^3.$$

Similar calculations are applied for all polynomials $f_r^{(1,2)}(x,y)$. Here they are:
$f_1^{(1,2)}(x,y) = x$,
$f_2^{(1,2)}(x,y) = y$,
$f_3^{(1,2)}(x,y) = xy$,
$f_4^{(1,2)}(x,y) = 3xy + 4x^2y$,
$f_5^{(1,2)}(x,y) = 3xy + 4xy^2$,
$f_6^{(1,2)}(x,y) = 5xy + 3x^2y + 6x^3y$,
$f_7^{(1,2)}(x,y) = 2xy + 5xy^2 + 5x^2y + 2x^2y^2$,
$f_8^{(1,2)}(x,y) = 5xy + 3xy^2 + 6xy^3$.
Thus,

$$a_2^y\, a_1^x = (x,\, y,\, xy,\, 3xy + 4x^2y,\, 3xy + 4xy^2, 5xy + 3x^2y + 6x^3y,\\ 2xy + 5xy^2 + 5x^2y + 2x^2y^2, 5xy + 3xy^2 + 6xy^3).$$

Using this method let's calculate other noncommutative $a_j^y\, a_i^x$.

$$a_3^y\, a_1^x = (x, 0, y, xy, 0, 3xy + 4x^2y, 0, 0),$$

$$a_3^y\, a_2^x = (0, x, y, 0, xy, 0, 0, 3xy + 4x^2y),$$
$$a_4^y\, a_1^x = (x, 0, 0, y, 0, xy, 0, 0),$$
$$a_4^y\, a_2^x = (0, x, 0, y, 0, 0, xy, 0),$$
$$a_5^y\, a_1^x = (x, 0, 0, 0, y, 0, xy, 0),$$
$$a_5^y\, a_2^x = (0, x, 0, 0, y, 0, 0, xy).$$

Not listed pairs are commutative, i.e. $a_j^y\, a_i^x = a_i^x\, a_j^y$.

Thus, we have a complete set of relations for the implementation of the collection process in analytical form:

$$a_j^y\, a_i^x = a_i^x\, a_j^y\, a_{j+1}^{f_{j+1}^{(i,j)}(x,y)}\, a_{j+2}^{f_{j+2}^{(i,j)}(x,y)} \dots a_8^{f_8^{(i,j)}(x,y)},\ 1 \leqslant i < j \leqslant 8. \tag{10}$$

Using (10) we can calculate the product $a_1^{x_1} \dots a_8^{x_8} \cdot a_1^{y_1} \dots a_8^{y_8} = a_1^{z_1} \dots a_8^{z_8}$. Following this procedure we will find all z_i (1–8).

The theorem is proved.

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Полиномы Холла конечных двупорожденных групп периода семь

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Пусть $B_k = B_0(2, 7, k)$ — максимальная конечная двупорожденная бернсайдова группа периода 7 ступени нильпотентности k. В настоящей статье вычислены полиномы Холла для B_k при $k \leqslant 4$.

Ключевые слова: периодическая группа, собирательный процесс, полиномы Холла.