On Solvability of the Cauchy Problem for a Loaded System

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In this work we investigated the Cauchy problem for a loaded Burgers-type system. Example of mathematical physics inverse problem leading to problem being investigated is given. Sufficient conditions for existence of solution in continuously differentiable class are obtained.

Keywords: Cauchy problem, inverse problem, Burgers’ equation, non-linear system, weak approximation method.

Inverse problems of mathematical physics play important role in science and applications today [1]. Coefficient inverse problems for parabolic equations are problems of finding solutions of differential equation with one (or more) unknown coefficients. These problems often reduce to problems for loaded equations. Loaded differential equations (see [2]) are ones with functionals of solution (e.g. values of solution or its derivatives on lesser-dimensional manifolds) as coefficients or right-hand side.

Existence of solution to special class of loaded two-dimensional parabolic equations has been proved by I. V. Frolenkov and Yu. Ya. Belov (see [3]). Problem being considered in this article arises during generalization of preceding results.

1. Problem formulation

We consider the initial-value problem for loaded system

\[ \frac{\partial \bar{u}}{\partial t} = \mu(t, \bar{\omega}(t)) \Delta \bar{u} + \nu(\bar{u} \cdot \nabla) \bar{u} + \bar{f}(t, x, \bar{u}, \bar{\omega}(t)), \]

\[ \bar{u}(0, x) = \bar{\varphi}(x) \]

in domain \( \Pi_{[0,T]} = \{(t,x)|0 \leq t \leq T, x \in \mathbb{R}^n \} \), where \( \bar{u} = (u_1(t,x), \ldots, u_n(t,x)) \) are unknown functions. Let \( \bar{\omega}(t) = (\bar{u}_i(t,x^j), D^\alpha u_i(t,x^j)) ; i = 1, \ldots, n; j = 1, \ldots, r; |\alpha| = 0, \ldots, p_0 \) be a vector function, with traces of unknown functions and their partial derivatives with respect to spatial variables of order up to \( p_0 \) at points \( x^1, \ldots, x^r \in \mathbb{R}^n \) as its components.

\[ D^\alpha = \frac{\partial^{\alpha}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \]

is partial differential operator, where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

Functions \( \mu(t, \bar{\omega}(t)), \bar{f} = (f_1, \ldots, f_n), \bar{\varphi} = (\bar{\varphi}_1(x), \ldots, \bar{\varphi}_n(x)) \) are given ones, \( \nu \in \mathbb{R} \) is given coefficient.

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We will use following notation:

\[
C^{q,a}(\Pi_{[0,T]}) = \left\{ \bar{u} = (u_1(t,x), \ldots, u_n(t,x)) \ \bigg| \ \frac{\partial^j u_i}{\partial \theta^j} : D^a u_i(t,x) \in C(\Pi_{[0,T]}); \right. \\
\left. \left| \frac{\partial^j u_i}{\partial \theta^j} \right| \leq K, |D^a u_i(t,x)| \leq K; \ i = 1, \ldots, n; \ j \leq q; \ |\alpha| \leq s; \ q,s \in \mathbb{Z}; \ K \text{ is const} \right\}
\]

is class of bounded, continuously differentiable functions,

\[
U^i_\alpha(0) = \sup_{x \in \mathbb{R}^n} |D^\alpha \varphi_i(x)|, \\
U^i_\alpha(t) = \sup_{\xi \in [0,t]} \sup_{x \in \mathbb{R}^n} |D^\alpha u_i(\xi, x)|, \\
U^i(t) = \max_{|\alpha| \leq p^2} U^i_\alpha(t), \ U(t) = 1 + \sum_{i=1}^n U^i(t)
\]

are nondecreasing nonnegative functions.

Let \( p \geq \max(p_0, 2) \), function \( \varphi \) satisfies

\[
\varphi_i(x) \in C^{p+2}(\mathbb{R}^n), \ |D^\alpha \varphi_i(x)| \leq K_1; \ i = 1, \ldots, n; \ |\alpha| \leq p + 2,
\]

\( \mu \) and \( f \) are continuous in all variables and the following relations are valid for any function \( \bar{u}(t,x) \in C^{1,p+2}(\Pi_{[0,T]}) \):

\[
\mu(t, \bar{u}(t,x)) \geq \mu_0 > 0, \ \forall \bar{u}(t,x) \in C^{1,p+2}(\Pi_{[0,T]})
\]

\[
|D^\alpha f_i(t,x, \bar{u}, \bar{\varphi})| \leq K_2(1 + U(t) + U(t)^2), \ |\alpha| \leq p + 2.
\]

Here and further, \( K_i \) are constants depending only on the initial data. We will prove

**Theorem 1.1.** Let the initial data of problem (1), (2) satisfy (3), (4) for some \( p \). Then constant \( t^* \) exists (\( t^* \in (0,T] \)) for which a solution of problem (1), (2) exists and lies in \( C^{1,p}(\Pi_{[0,t^*]}) \) class.

2. **An example**

We have investigated inverse problem involving finding functions \( u(t,x), g(t) \) in Cauchy problem for the Burgers-type equation

\[
u_t(t,x) = \mu(t) u_{xx} + A(t) u u_x + B(t) u + g(t) f(t,x), \\
u(0,x) = u_0(x), \quad 0 \leq t \leq T, \quad x \in \mathbb{R},
\]

which reduces (using overdetermination condition \( u(t,x_0) = \bar{u}(t) \)) to Cauchy problem for loaded parabolic equation

\[
u_t(t,x) = \mu(t) u_{xx} + A(t) u u_x + B(t) u + F(t,u), \\
u(0,x) = u_0(x), \quad 0 \leq t \leq T, \quad x \in \mathbb{R},
\]

where

\[
F(t,u) = \frac{f(t,x)}{f(t,x_0)} (\psi^r(t) - B(t) \psi(t) - \mu(t) u_{xx}(t,x_0) - A(t) \psi(t) u_x(t,x_0))
\]

is functional depending on traces of unknown function and its derivatives at point \( x_0 \). The problem (5), (6) is the particular case of problem (1), (2) for \( n = 1, \bar{u} = u(t,x), \bar{f} = F(t,u), \varphi = u_0(x) \).
Let the initial data of problem the (5), (6) satisfies
\[ u_0(x) \in C^{p+2}(\mathbb{R}), \quad \left| \frac{\partial^k u_0}{\partial x^k} \right| \leq K_0 - \text{const}, \quad k = 0, \ldots, p + 2, \] (7)

\[ A(t), B(t) \in C([0, T]), \quad \psi(t) \in C^1([0, T]), \quad |f(t, x_0)| \geq \frac{1}{K_0}, \]
\[ |A(t)| + |B(t)| + |\psi(t)| + |\psi'(t)| \leq K_0, \]
\[ \mu(t) \geq \mu_0 > 0, \quad \frac{\partial^k f}{\partial x^k} \in C([0, T] \times \mathbb{R}), \quad \left| \frac{\partial^k f}{\partial x^k} \right| \leq K_0, \quad k = 0, \ldots, p + 2 \] (8)

for some \( p \geq 2 \). Conditions (3) of Theorem 1.1 are fulfilled by (7). We can check fulfillment of (3) provided (8) are valid:

\[ \forall u(t, x) \in C^{1, p+2}([0, T] \times \mathbb{R}) \quad \forall k = 0, \ldots, p + 2 \quad \left| \frac{\partial^k}{\partial x^k} F(t, u) \right| \leq K_2 \left( 1 + U^1_2(t) + U^1_1(t) \right) \leq K_2 (1 + U(t)). \]

Thus in is the particular case one can use Theorem 1.1 to prove existence of solution of problem (5), (6) in \( C^{1, p}(\Pi_{[0, T])} \) class.

3. Auxiliary theorem

**Theorem 3.1.** Let \( u(t, x) \) be solution of

\[ u_t = \sum_{i=1}^{n} b_i(t, x) \frac{\partial u(t, x)}{\partial x_i}, \]
\[ u(0, x) = u_0(x), \quad x \in E_n \]

in domain \( G_{[0, T]} = \{ (t, x)| 0 \leq t \leq T, x \in E_n \} \) of \( C^{1, p}(G_{[0, T]}) \) class. Let the conditions

\[ |D^\alpha b_i(t, x)| \leq M(p), \quad |\alpha| \leq p, i = 1, \ldots, n; \]
\[ |D^\alpha u_0(x)| \leq C(p), \quad |\alpha| \leq p \]

are valid. Then \( u(t, x) \) satisfies

\[ |D^\alpha u(t, x)| \leq C(p)e^{l(p)M(p)T}, \quad |\alpha| \leq p, \] (9)

where \( l(p) > 0 \) depends only on \( p \) and does not depend on the initial data.

4. Proof of Theorem 1.1

We will prove existence of solution of problem (1), (2) using weak approximation method (see [4]). We split the problem into three fractional steps and make time shift by \( \tau/3 \) in traces.
of unknown functions and nonlinear terms. This leads to equation system

\[
\frac{\partial u^\alpha_t}{\partial t} = 3\mu(t, \omega(t - \tau / 3)) \Delta u^\alpha_t, \quad t \in (m\tau, (m + 1 / 3)\tau],
\]

\[
\frac{\partial u^\alpha_t}{\partial t} = 3\nu(\bar{u}^\alpha(t - \tau / 3) \cdot \nabla) u^\alpha_t, \quad t \in ((m + 1 / 3)\tau, (m + 2 / 3)\tau],
\]

\[
\frac{\partial u^\alpha_t}{\partial t} = 3f_i(t - \tau / 3, x, \bar{u}^\alpha(t - \tau / 3, x), \omega(t - \tau / 3)),
\]

\[
t \in ((m + 2 / 3)\tau, (m + 1)\tau],
\]

\[
|u^\alpha_t(t, x)|_{t \leq \tau} = \varphi_i(x); \quad i = 1, \ldots, n; \quad m = 0, \ldots, M - 1; \quad M\tau = T.
\]

Let us introduce the following notation

\[
U^{\alpha\tau}_i(t) = \sup_{\xi \in [0, t]} \sup_{x \in \mathbb{R}^n} |D^{\alpha} u^\alpha_i(\xi, x)|,
\]

\[
U^{\tau}(t) = \max_{|\alpha| \leq p + 2} U^{\alpha\tau}_i(t), \quad U^{\tau}(t) = 1 + \sum_{i=1}^{n} U^{\alpha\tau}_i(t).
\]

Zeroth whole step \((m = 0)\) is considered. In first fractional step system (10), (13) is representing \(n\) Cauchy problems for parabolic equations, for which the maximum principle can be applied. We differentiate (10), (13) with respect to spartial variables up to \((p + 2)\) times, thus obtaining

\[
U^{\alpha\tau}_i(t) \leq U^{\alpha\tau}_i(0), \quad U^{\tau}(t) \leq U(0), \quad |\alpha| \leq p + 2, \quad t \in (0, \tau / 3].
\]

In second fractional step (11), (13) is \(n\) separate linear first-order partial differential equations

\[
\frac{\partial u^\alpha_i}{\partial t} = 3\nu u^\alpha_i(t - \tau / 3, x) \frac{\partial u^\alpha_i}{\partial x_1} + \cdots + 3\nu u^\alpha_i(t - \tau / 3, x) \frac{\partial u^\alpha_i}{\partial x_n},
\]

\[
u u^\alpha_i|_{t = \tau / 3} = u^\alpha_i(\tau / 3, x), \quad i = 1, \ldots, n,
\]
solutions of which satisfy Theorem 3.1, giving us estimate (with \(K_3\) equals to \(l(p + 2)\) arising in Theorem 3.1)

\[
|D^{\alpha} u^\alpha_i(t, x)| \leq U^{\alpha\tau}(\tau / 3)e^{K_3 U^{\tau}(\tau / 3)}, \quad |\alpha| \leq p + 2, \quad t \in (\tau / 3, 2\tau / 3],
\]

leading to

\[
U^{\tau}(t) \leq U^{\tau}(\tau / 3)e^{K_3 U^{\tau}(\tau / 3)}, \quad t \in (\tau / 3, 2\tau / 3].
\]

In third fractional step \(u^\alpha_i(t, x)\) are solutions to \(n\) separate Cauchy problems for ordinary differential equations with known right-hand sides. Thus \(u^\alpha_i(t, x)\) and their derivatives can be expressed explicitly

\[
D^{\alpha} u^\alpha_i(t, x) = D^{\alpha} u^\alpha_i(\frac{2\tau}{3}, x) + \int_{\tau/3}^{t} 3D^{\alpha} f_i\left(\xi - \frac{\tau}{3}, x, \bar{u}^\alpha(\xi - \frac{\tau}{3}, x), \omega(\xi - \frac{\tau}{3})\right) d\xi,
\]

\[
|\alpha| \leq p + 2, \quad t \in (\frac{2\tau}{3}, \tau],
\]

and using (4) can be estimated\(^1\) by

\[
U^{\tau}(t) \leq U^{\tau}(\frac{2\tau}{3})e^{K_3 U^{\tau}(\frac{2\tau}{3})}, \quad t \in (\frac{2\tau}{3}, \tau].
\]

\(^1\)For detailed derivation of (16), see Appendix 5.
Let \( t^* \) be nonnegative constant satisfying
\[
e^{6t^*K_5U(0)} \leq 2, \quad K_5 = \max(K_3, K_4).
\] (17)

We will prove that derivatives \( \{D^\alpha u^\tau\}, |\alpha| \leq p + 2 \) are bounded uniformly on \( \tau \) in some time interval \( 0 \leq t \leq t^* \). Here and further \( \tau \) be arbitrary small \( (\tau \ll t^*) \) and for some integer \( M' = M'(\tau) \) equality \( M'\tau = t^* \) is valid. From (17)
\[
e^{(2i-1)3\tau K_5U(0)} \leq 2, \quad i = 1, \ldots, M'.
\] (18)

Using (18) we express from (14)–(16) estimate valid in \( t \in [0, \tau] \)
\[
U^\tau(t) \leq U(0)e^{3\tau K_5U(0)}.
\] (19)

We will prove the inequality
\[
U^\tau(i\tau) \leq U(0)\exp((2i-1)3\tau K_5U(0)) = K_6, \quad i = 1, \ldots, M',
\] (20)

by induction. For \( i = 1 \) (20) is valid by (19). Let (20) be valid for some \( i < M' \). Applying our reasoning as in zeroth whole step, we deduce
\[
U^\tau((i+1)\tau) \leq U^\tau(i\tau)e^{3\tau K_5U^\tau(i\tau)} \leq U(0)\exp((2i-1)3\tau K_5U(0))\exp(3\tau K_5U(0)e^{(2i-1)3\tau K_5U(0)}) \leq U(0)\exp((2i+1)3\tau K_5U(0)) = U(0)\exp((2i+1) - 3\tau K_5U(0)),
\]
thus validating (20) for \( i + 1 \). It holds for all \( i < M' \) by mathematical induction principle.

Since \( U^\tau(t) \) is monotonic, from (20) we have
\[
U^\tau(t) \leq U^\tau(M'\tau) = K_6 - \text{const}, \quad t \in [0, t^*].
\]

From the previous inequality it follows that uniform on \( \tau \)
\[
|D^\alpha u^\tau(t, x)| \leq K_6, (t, x) \in \Pi_{[0, 1]}, |\alpha| \leq p + 2,
\] (21)

where \( \Pi_{[0, 1]} = \{(t, x)|0 \leq t \leq t^*, x \in \mathbb{R}^n\} \).

Derivatives
\[
\frac{\partial}{\partial t} D^\alpha\bar{u}^\tau(t, x), \frac{\partial}{\partial x_i} D^\alpha\bar{u}^\tau(t, x), \quad (t, x) \in \Pi_{M_0}^{M_0} \cup \Pi_{[0, 1]}, |\alpha| \leq p, \quad i = 1, \ldots, n,
\]

where \( \Pi_{M_0}^{M_0} = \{(t, x), t \in [0, t^*], |x| \leq M_0\} \), are bounded uniformly on \( \tau \) from (21) and equations (10)–(12), which implies uniform boundedness and uniform equicontinuity (for any \( M_0 > 0 \)) of function sets \( \{D^\alpha\bar{u}^\tau\}, |\alpha| \leq p \) in \( \Pi_{[0, 1]}^{M_0} \).

Applying Arzelà–Ascoli theorem about compactness, we show existence of the subsequence \( \bar{u}^\tau(t, x) \) of sequence \( \bar{u}^\tau(t, x) \), which converges to some vector function \( \bar{u}(t, x) \) with its derivatives \( D^\alpha\bar{u}(t, x) \), \( |\alpha| \leq p \). Under the theorem about weak approximation method convergence [4] the vector function \( \bar{u}(t, x) \) is a solution (of \( C^{1, p}(\Pi_{[0, 1]}^{M_0}) \) class) to (1), (2) in \( |x| \leq M_0 \), and
\[
\|D^\alpha\bar{u}^\tau - D^\alpha\bar{u}\|_{C^{1, p}(\Pi_{[0, 1]}^{M_0})} \to 0, \quad |\alpha| \leq p
\]
for \( \tau \to 0 \).

Since \( M_0 \) is arbitrary constant, the vector function \( \bar{u}(t, x) \) is a solution to (1), (2) in whole \( \Pi_{[0, 1]} \) domain. Theorem 1.1 proved.
5. Derivation of inequality (15)

We are given with

\[ D^\alpha u^\tau_i(t, x) = D^\alpha u^\tau_i \left( \frac{2\tau}{3}, x \right) + \int_{\frac{2\tau}{3}}^{t} 3D^\alpha f_i \left( \xi - \frac{\tau}{3}, x, \bar{u}^\tau(\xi - \frac{\tau}{3}, x), \bar{\omega}(\xi - \frac{\tau}{3}) \right) d\xi, \]

\[ \alpha \leq p + 2, \ t \in (\frac{2\tau}{3}, \tau], \]

Taking absolute value of both sides of the previous equality and using (4) we have

\[ |D^\alpha u^\tau_i(t, x)| \leq |D^\alpha u^\tau_i \left( \frac{2\tau}{3}, x \right)| + \int_{\frac{2\tau}{3}}^{t} 3K_2 \left( 1 + U^\tau(\xi - \frac{\tau}{3}) + U^\tau(\xi - \frac{\tau}{3})^2 \right) d\xi. \]

Since \( \frac{2\tau}{3} \leq \xi \leq t \leq \tau \) and \( U(t) \) is nondecreasing function, it is true that \( U(\xi - \frac{\tau}{3}) \leq U(\frac{2\tau}{3}) \):

\[ |D^\alpha u^\tau_i(t, x)| \leq |D^\alpha u^\tau_i \left( \frac{2\tau}{3}, x \right)| + \int_{\frac{2\tau}{3}}^{t} 3K_2 \left( 1 + U^\tau(\frac{2\tau}{3}) + U^\tau(\frac{2\tau}{3})^2 \right) d\xi. \]

Integrand in the previous inequality does not depend on the integration variable. \( \int_{\frac{2\tau}{3}}^{t} d\xi \leq \tau / 3. \)

As \( U^\tau(t) \geq 1 \), it is obvious that \( U^\tau(\frac{2\tau}{3})^2 \geq U^\tau(\frac{2\tau}{3}) \geq 1. \) Thus

\[ |D^\alpha u^\tau_i(t, x)| \leq |D^\alpha u^\tau_i \left( \frac{2\tau}{3}, x \right)| + 3\tau K_2 U^\tau(\frac{2\tau}{3})^2. \]

We apply \( \sup_{x \in \mathbb{R}^n} \) first, then \( \sup_{(0, \ell]} \) to both parts of the previous inequality:

\[ U^\tau_\alpha(t) \leq U^\tau_\alpha \left( \frac{2\tau}{3} \right) + 3\tau K_2 U^\tau \left( \frac{2\tau}{3} \right)^2. \]

Taking \( \max_\alpha \) for \( |\alpha| \leq p + 2 \), and calculating sum for \( i = 1, \ldots, n \), we obtain

\[ U^\tau(t) \leq U^\tau \left( \frac{2\tau}{3} \right) + 3\tau K_2 U^\tau \left( \frac{2\tau}{3} \right)^2. \]

Let \( K_2 \) be equal \( 3aK_2 \). We factor out \( U^\tau \left( \frac{2\tau}{3} \right) \):

\[ U^\tau(t) \leq U^\tau \left( \frac{2\tau}{3} \right) \cdot \left( 1 + \tau K_2 U^\tau \left( \frac{2\tau}{3} \right) \right). \]

Using \( 1 + x \leq e^x \) we finally get

\[ U^\tau(t) \leq U^\tau \left( \frac{2\tau}{3} \right) \cdot e^{\tau K_2 U^\tau \left( \frac{2\tau}{3} \right)}. \]

References


О разрешимости задачи Коши для системы нагруженных уравнений

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В работе рассмотрена задача Коши для системы нагруженных уравнений типа Бюргерса. Приведен пример обратной задачи математической физики, сводящейся к рассматриваемой задаче. Получены достаточные условия существования решения задачи в классе гладких ограниченных функций.

Ключевые слова: задача Коши, обратные задачи, уравнение Бюргерса, система нелинейных уравнений, метод слабой аппроксимации.