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Group Analysis of Three-dimensional Equations of an Ideal Fluid in Terms of Trajectories and Weber Potential

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Lie group analysis of equations of an ideal fluid written in variables of trajectories and Weber's potential was conducted. It was shown that the use of volume conserving arbitrary Lagrangian coordinates is in fact an equivalent transformation for the equations. The defining Lie algebra equations for the initial velocity distribution were obtained. The basic Lie group and its extensions were found.

Keywords: equations of an ideal fluid, equivalent transformation, Lagrangian coordinates, defining equations.

Introduction

In describing the motion of an ideal incompressible fluid with a free boundary one needs to find the solution of the Euler equations subject to kinematic and dynamic conditions at the free boundary. The kinematic condition allows us to transform the initial problem to the problem with the fixed domain. This is achieved with the use of the Lagrangian coordinates $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ which are the coordinates of fluid particles at the initial point in time $t = 0$: $\mathbf{x} = \boldsymbol{\xi}$. The particle coordinates $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$ are defined by the equation $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$.

A system of equations of the following type is considered [1]

$$x_t = (y_\eta z_\zeta - z_\eta y_\zeta)(\varphi_\xi - u_0) + (-y_\xi z_\zeta + z_\xi y_\zeta)(\varphi_\eta - v_0) + (y_\xi z_\eta - z_\xi y_\eta)(\varphi_\zeta - w_0), \qquad (0.1)$$

$$y_t = (-x_\eta z_\zeta + z_\eta x_\zeta)(\varphi_\xi - u_0) + (x_\xi z_\zeta - z_\xi x_\zeta)(\varphi_\eta - v_0) + (-x_\xi z_\eta + z_\xi x_\eta)(\varphi_\zeta - w_0), \qquad (0.2)$$

$$z_t = (x_\eta y_\zeta - y_\eta x_\zeta)(\varphi_\xi - u_0) + (-x_\xi y_\zeta + y_\xi x_\zeta)(\varphi_\eta - v_0) + (x_\xi y_\eta - y_\xi x_\eta)(\varphi_\zeta - w_0), \qquad (0.3)$$

$$x_\xi(y_\eta z_\zeta - y_\zeta z_\eta) + x_\eta(-y_\xi z_\zeta + y_\zeta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi) = 1, \qquad (0.4)$$

where $(x, y, z) = \mathbf{x}(\boldsymbol{\xi}, t)$ are the fluid particles coordinates, $\varphi((\boldsymbol{\xi}, t))$ is the required function that arises from the transformation of the equations of motion and $u_0(\xi, \eta, \zeta)$, $v_0(\xi, \eta, \zeta)$, $w_0(\xi, \eta, \zeta)$ are the components of the particle velocity vector at $t = 0$. The transformation of equations of motion with respect to variables $\mathbf{x}$ and φ was first discovered by G. Weber [2]. Equation (0.4) describes the volume conservation, $detM = 1$, where $M = \partial(\mathbf{x})/\partial(\boldsymbol{\xi})$ is the Jacobi matrix.

To study the group properties of equations (0.1)–(0.4) the following index designations are introduced

$$x^1 = \xi, \ x^2 = \eta, \ x^3 = \zeta, \ x^4 = t,$$
$$u^1 = x, \ u^2 = y, \ u^3 = z, \ u^4 = \varphi,$$

$u^k = u^k(\xi, \ \eta, \ \zeta, \ t), \ \ k = \overline{1, \ 4}.$

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Let us rewrite the system of equations using the index designations

$$u_4^1 = \left(u_2^2 u_3^3 - u_2^3 u_3^2\right)\left(u_1^4 + u_0\right) + \left(-u_1^2 u_3^3 + u_1^3 u_3^2\right)\left(u_2^4 + v_0\right) + \left(u_1^2 u_2^3 - u_1^3 u_2^2\right)\left(u_3^4 + w_0\right), \quad (0.5)$$

$$u_4^2 = \left(-u_2^1 u_3^3 + u_2^3 u_3^1\right)\left(u_1^4 + u_0\right) + \left(u_1^1 u_3^3 - u_1^3 u_3^1\right)\left(u_2^4 + v_0\right) + \left(-u_1^1 u_2^3 + u_1^3 u_2^1\right)\left(u_3^4 + w_0\right), \quad (0.6)$$

$$u_4^3 = \left(u_2^1 u_3^2 - u_2^2 u_3^1\right)\left(u_1^4 + u_0\right) + \left(-u_1^1 u_3^2 + u_1^2 u_3^1\right)\left(u_2^4 + v_0\right) + \left(u_1^1 u_2^2 - u_1^2 u_2^1\right)\left(u_3^4 + w_0\right), \quad (0.7)$$

$$u_1^1\left(u_2^2 u_3^3 - u_2^3 u_3^2\right) + u_2^1\left(-u_1^2 u_3^3 + u_1^3 u_3^2\right) + u_3^1\left(u_1^2 u_2^3 - u_1^3 u_2^2\right) - 1 = 0. \quad (0.8)$$

Here u_0, v_0, w_0 are the functions of the initial velocity distribution at $t = 0$. They depend on (x^1, x^2, x^3) and are found to be the defining functions for the given system.

It can be shown that transition to arbitrary Lagrangian coordinates $(\alpha, \beta, \gamma) = \boldsymbol{\alpha}(\boldsymbol{\xi})$ which conserves the volume ($det J = 1$, where $J = \partial(\alpha, \beta, \gamma)/\partial(\xi, \eta, \zeta)$ is the Jacobi matrix) is the equivalent transformation for equations (0.1)–(0.4). The structure of equations (0.1)–(0.4) is not changed after such transformation. The components of the initial velocity vector are changed and they are described by the following formulas

$$U_0 = (\beta_\eta \gamma_\zeta - \beta_\zeta \gamma_\eta)u_0^1 - (\beta_\xi \gamma_\zeta - \gamma_\xi \beta_\zeta)v_0^1 + (\beta_\xi \gamma_\eta - \beta_\eta \gamma_\xi)w_0^1,$$

$$V_0 = (\gamma_\eta \alpha_\zeta - \gamma_\zeta \alpha_\eta)u_0^1 - (\gamma_\xi \alpha_\zeta - \alpha_\xi \gamma_\zeta)v_0^1 + (\gamma_\xi \alpha_\eta - \gamma_\eta \alpha_\xi)w_0^1,$$

$$W_0 = (\alpha_\eta \beta_\zeta - \alpha_\zeta \beta_\eta)u_0^1 - (\alpha_\xi \beta_\zeta - \beta_\xi \alpha_\zeta)v_0^1 + (\alpha_\xi \beta_\eta - \alpha_\eta \beta_\xi)w_0^1,$$

where $(u_0^1, v_0^1, w_0^1) = \mathbf{u_0}\left(\boldsymbol{\xi}(\alpha, \beta, \gamma)\right)$ and $div\mathbf{u_0} = 0$.

1. Formulation of the problem and group analysis of equations

It is necessary to find the kernel of basic Lie algebra of the transformation of system (0.1)–(0.4) and all specifications of the elements u_0, v_0, w_0 that give us an extension of the Lie algebra [3].

We define the infinitesimal operator for system (0.5)–(0.8) in the following way

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k},$$

$i = \overline{1, 4}$, $k = \overline{1, 4}$. From this point on we assume summation for all repeating indices. We assume that elements (ξ^1, ξ^2, ξ^3) depend on (x^1, x^2, x^3) and ξ^4 depends only on x^4. We also assume that $(\eta^1, \eta^2, \eta^3, \eta^4)$ depend on $(\xi^1, \xi^2, \xi^3, \xi^4, u^1, u^2, u^3, u^4)$.

System of equations (0.5)–(0.8) has first order derivatives. To construct the determining equations it is necessary to extend the operator X to the first order derivatives

$$\underset{1}{X} = X + \zeta_i^k \frac{\partial}{\partial u_i^k}, \quad \zeta_i^k = \frac{\partial \eta^k}{\partial x^i} + u_i^n \frac{\partial \eta^k}{\partial u^n} - u_j^k \left(\frac{\partial \xi^j}{\partial x^i} + u_i^n \frac{\partial \xi^j}{\partial u^n}\right),$$

$n = \overline{1, 4}$, $j = \overline{1, 4}$.

Let us use the criterion of invariance [3] when the action of the operator $\underset{1}{X}$ on equations (0.5)–(0.8) gives us zero. It means transition to a manifold of equations (0.5)–(0.8). By expressing elements u_4^1, u_4^2 and u_4^3 from equations (0.5)–(0.7) in terms of the remaining variables we determine the manifold M. We express element u_1^1 from equation (0.8) in the following way

$$u_1^1 = \left(u_2^2 u_3^3 - u_2^3 u_3^2\right)^{-1} + u_2^1\left(u_1^2 u_3^3 - u_1^3 u_3^2\right)\left(u_2^2 u_3^3 - u_2^3 u_3^2\right)^{-1} - u_3^1\left(u_1^2 u_2^3 - u_1^3 u_2^2\right)\left(u_2^2 u_3^3 - u_2^3 u_3^2\right)^{-1}.$$

Let us consider each of the equations.

The result of action of the operator on equation (0.8) gives

$$\underset{1}{X}\left(u_1^1\left(u_2^2u_3^3 - u_2^3u_3^2\right) + u_2^1\left(-u_1^2u_3^3 + u_1^3u_3^2\right) + u_3^1\left(u_1^2u_2^3 - u_1^3u_2^2\right) - 1\right) = 0.$$

The extended version of the previous expression is

$$\zeta_1^1\left(u_2^2u_3^3 - u_2^3u_3^2\right) + u_1^1\left(\zeta_2^2u_3^3 + u_2^2\zeta_3^3 - \zeta_2^3u_3^2 - u_2^3\zeta_3^2\right) - \zeta_2^1\left(u_1^2u_3^3 - u_1^3u_3^2\right) -$$
$$-u_2^1\left(\zeta_1^2u_3^3 + u_1^2\zeta_3^3 - \zeta_1^3u_3^2 - u_1^3\zeta_3^2\right) + \zeta_3^1\left(u_1^2u_2^3 - u_1^3u_2^2\right) + u_3^1\left(\zeta_1^2u_2^3 + u_1^2\zeta_2^3 - \zeta_1^3u_2^2 - u_1^3\zeta_2^2\right) = 0.$$

Let us switch to manifold M which is defined by equations (0.1)–(0.5). In what follows we split the resulting equation with respect to the independent variables u_j^i, $i, j = \overline{1, 4}$ (we exclude variables u_4^1, u_4^2, u_4^3, u_1^1). The results of the splitting are presented below.

We notice that the derivatives of coordinates of the operator X are equal to zero for several variables, namely,

$$\frac{\partial \eta^1}{\partial u^4} = \frac{\partial \eta^2}{\partial u^4} = \frac{\partial \eta^3}{\partial u^4} = 0, \quad \frac{\partial \eta^1}{\partial x^1} = \frac{\partial \eta^1}{\partial x^2} = \frac{\partial \eta^1}{\partial x^3} = 0, \tag{1.1}$$

$$\frac{\partial \eta^2}{\partial x^1} = \frac{\partial \eta^2}{\partial x^2} = \frac{\partial \eta^2}{\partial x^3} = 0, \quad \frac{\partial \eta^3}{\partial x^1} = \frac{\partial \eta^3}{\partial x^2} = \frac{\partial \eta^3}{\partial x^3} = 0. \tag{1.2}$$

The following equation for the coordinates is valid

$$\frac{\partial \eta^1}{\partial u^1} + \frac{\partial \eta^2}{\partial u^2} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} = 0. \tag{1.3}$$

Let us analyze equation (0.5). The result of action of the extended operator on this equation with respect to the coordinates used in equations (1.1)–(1.2) is

$$\underset{1}{X}\left(\left(u_2^2u_3^3 - u_2^3u_3^2\right)\left(u_1^4 + u_0\right) + \left(-u_1^2u_3^3 + u_1^3u_3^2\right)\left(u_2^4 + v_0\right) + \left(u_1^2u_2^3 - u_1^3u_2^2\right)\left(u_3^4 + w_0\right) - u_4^1\right) = 0.$$

The extended version of the previous expression is

$$-\zeta_4^1 + \left(\zeta_2^2u_3^3 + u_2^2\zeta_3^3 - \zeta_2^3u_3^2 - u_2^3\zeta_3^2\right)\left(u_1^4 + u_0\right) + \left(u_2^2u_3^3 - u_2^3u_3^2\right)\left(\zeta_1^4 + \xi^1\frac{\partial u_0}{\partial x^1} + \xi^2\frac{\partial u_0}{\partial x^2} + \xi^3\frac{\partial u_0}{\partial x^3}\right) +$$

$$+ \left(-\zeta_1^2u_3^3 - u_1^2\zeta_3^3 + \zeta_1^3u_3^2 + u_1^3\zeta_3^2\right)\left(u_2^4 + v_0\right) + \left(-u_1^2u_3^3 + u_1^3u_3^2\right)\left(\zeta_2^4 + \xi^1\frac{\partial v_0}{\partial x^1} + \xi^2\frac{\partial v_0}{\partial x^2} + \xi^3\frac{\partial v_0}{\partial x^3}\right) +$$

$$+ \left(\zeta_1^2u_2^3 + u_1^2\zeta_2^3 - \zeta_1^3u_2^2 - u_1^3\zeta_2^2\right)\left(u_3^4 + w_0\right) + \left(u_1^2u_2^3 - u_1^3u_2^2\right)\left(\zeta_3^4 + \xi^1\frac{\partial w_0}{\partial x^1} + \xi^2\frac{\partial w_0}{\partial x^2} + \xi^3\frac{\partial w_0}{\partial x^3}\right) = 0.$$

We obtain equation in which the transition to manifold M is realized. The resulting system of equations is split with respect to independent variables shown above. The following conclusions are made as a result of the splitting of the equation. First of all, we found out that

$$\frac{\partial \eta^4}{\partial u^1} = \frac{\partial \eta^1}{\partial x^4} = 0. \tag{1.4}$$

Second, we obtained the following type of equations

$$\frac{\partial \eta^4}{\partial u^4} + \frac{\partial \eta^3}{\partial u^3} + \frac{\partial \eta^2}{\partial u^2} - \frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} = 0, \tag{1.5}$$

$$v_0\left(\frac{\partial \eta^2}{\partial u^1} + \frac{\partial \eta^1}{\partial u^2}\right) = 0, \quad w_0\left(\frac{\partial \eta^2}{\partial u^1} + \frac{\partial \eta^1}{\partial u^2}\right) = 0, \tag{1.6}$$

$$v_0 \left(\frac{\partial \eta^3}{\partial u^1} + \frac{\partial \eta^1}{\partial u^3} \right) = 0, \ w_0 \left(\frac{\partial \eta^3}{\partial u^1} + \frac{\partial \eta^1}{\partial u^3} \right) = 0. \tag{1.7}$$

From equations (1.6)–(1.7) four cases follow: 1) $v_0 \neq 0, \ w_0 \neq 0$; 2) $v_0 = 0, \ w_0 \neq 0$; 3) $v_0 \neq 0$, $w_0 = 0$; 4) $v_0 = 0; \ w_0 = 0$. The first three cases result in the following equations

$$\frac{\partial \eta^2}{\partial u^1} + \frac{\partial \eta^1}{\partial u^2} = 0, \ \frac{\partial \eta^3}{\partial u^1} + \frac{\partial \eta^1}{\partial u^3} = 0. \tag{1.8}$$

The last case is not taken into account.

Third, we obtained equation that contains components of the velocity vector $(u_0, \ v_0, \ w_0)$ in explicit form:

$$\frac{\partial \eta^4}{\partial x^1} + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} +$$
$$+ u_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} - \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + v_0 \frac{\partial \xi^2}{\partial x^1} + w_0 \frac{\partial \xi^3}{\partial x^1} = 0, \tag{1.9}$$

$$\frac{\partial \eta^4}{\partial x^2} + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} +$$
$$+ u_0 \frac{\partial \xi^1}{\partial x^2} + v_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} - \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + w_0 \frac{\partial \xi^3}{\partial x^2} = 0, \tag{1.10}$$

$$\frac{\partial \eta^4}{\partial x^3} + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} +$$
$$+ u_0 \frac{\partial \xi^1}{\partial x^3} + v_0 \frac{\partial \xi^2}{\partial x^3} + w_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) = 0. \tag{1.11}$$

Similarly to the previous case, the result of action of the extended operator on equation (0.6) is

$$-\zeta_4^2 + \left(-\zeta_2^1 u_3^3 - u_2^1 \zeta_3^3 + \zeta_2^3 u_3^1 + u_2^3 \zeta_3^1 \right) (u_1^4 + u_0) + \left(-u_2^1 u_3^3 + u_2^3 u_3^1 \right) \left(\zeta_1^4 + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} \right) +$$

$$+ \left(\zeta_1^1 u_3^3 + u_1^1 \zeta_3^3 - \zeta_1^3 u_3^1 - u_1^3 \zeta_3^1 \right) (u_2^4 + v_0) + \left(u_1^1 u_3^3 - u_1^3 u_3^1 \right) \left(\zeta_2^4 + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} \right) +$$

$$+ \left(-\zeta_1^1 u_2^3 - u_1^1 \zeta_2^3 + \zeta_1^3 u_2^1 + u_1^3 \zeta_2^1 \right) (u_3^4 + w_0) + \left(-u_1^1 u_2^3 + u_1^3 u_2^1 \right) \left(\zeta_3^4 + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} \right) = 0.$$

As a result of the splitting with respect to independent variables we obtained equation

$$\frac{\partial \eta^4}{\partial u^2} = \frac{\partial \eta^2}{\partial x^4} = 0 \tag{1.12}$$

and the following equations

$$\frac{\partial \eta^4}{\partial u^4} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \eta^2}{\partial u^2} + \frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} = 0, \tag{1.13}$$

$$\frac{\partial \eta^3}{\partial u^2} + \frac{\partial \eta^2}{\partial u^3} = 0, \ \ \frac{\partial \eta^4}{\partial u^2} + \frac{\partial \eta^2}{\partial x^4} = 0. \tag{1.14}$$

The last equations were obtained after considering the mentioned above cases: 1) $v_0 \neq 0, \ w_0 \neq 0$; 2) $v_0 = 0, \ w_0 \neq 0$; 3) $v_0 \neq 0, \ w_0 = 0$; 4) $v_0 = 0; \ w_0 = 0$.

We have also obtained the following equations

$$\frac{\partial \eta^4}{\partial x^1} + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} +$$

$$+ u_0 \left(-\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + v_0 \frac{\partial \xi^2}{\partial x^1} + w_0 \frac{\partial \xi^3}{\partial x^1} = 0, \qquad (1.15)$$

$$\frac{\partial \eta^4}{\partial x^2} + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} +$$

$$+ u_0 \frac{\partial \xi^1}{\partial x^2} + v_0 \left(-\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + w_0 \frac{\partial \xi^3}{\partial x^2} = 0, \qquad (1.16)$$

$$\frac{\partial \eta^4}{\partial x^3} + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} +$$

$$+ u_0 \frac{\partial \xi^1}{\partial x^3} + v_0 \frac{\partial \xi^2}{\partial x^3} + w_0 \left(-\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) = 0. \qquad (1.17)$$

Let us note that equations (1.15)–(1.17) differ from equations (1.9)–(1.11) by the terms in parentheses.

Finally, the result of action of the extended operator $\underset{1}{X}$ on equation (0.7) is

$$-\zeta_4^3 + \left(\zeta_2^1 u_3^2 + u_2^1 \zeta_3^2 - \zeta_2^2 u_3^1 - u_2^2 \zeta_3^1 \right)\left(u_1^4 + u_0 \right) + \left(u_2^1 u_3^2 - u_2^2 u_3^1 \right)\left(\zeta_1^4 + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} \right) +$$

$$+ \left(-\zeta_1^1 u_3^2 - u_1^1 \zeta_3^2 + \zeta_1^2 u_3^1 + u_1^2 \zeta_3^1 \right)\left(u_2^4 + v_0 \right) + \left(-u_1^1 u_3^2 + u_1^2 u_3^1 \right)\left(\zeta_2^4 + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} \right) +$$

$$+ \left(\zeta_1^1 u_2^2 + u_1^1 \zeta_2^2 - \zeta_1^2 u_2^1 - u_1^2 \zeta_2^1 \right)\left(u_3^4 + w_0 \right) + \left(u_1^1 u_2^2 - u_1^2 u_2^1 \right)\left(\zeta_3^4 + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} \right) = 0.$$

As in previous cases, the splitting of the equation with respect to independent variables leads to the following result

$$\frac{\partial \eta^4}{\partial u^3} = \frac{\partial \eta^3}{\partial x^4} = 0, \qquad (1.18)$$

$$\frac{\partial \eta^4}{\partial u^4} - \frac{\partial \eta^3}{\partial u^3} + \frac{\partial \eta^2}{\partial u^2} + \frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} = 0, \qquad (1.19)$$

$$\frac{\partial \eta^3}{\partial u^2} + \frac{\partial \eta^2}{\partial u^3} = 0, \quad \frac{\partial \eta^4}{\partial u^3} + \frac{\partial \eta^3}{\partial x^4} = 0, \qquad (1.20)$$

given that 1) $v_0 \neq 0$, $w_0 \neq 0$; 2) $v_0 = 0$, $w_0 \neq 0$; 3) $v_0 \neq 0$, $w_0 = 0$; 4) $v_0 = 0$; $w_0 = 0$.

Besides, there are equations which contain the coordinates of the operator and the components of initial velocity:

$$\frac{\partial \eta^4}{\partial x^1} + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} +$$

$$+ u_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + v_0 \frac{\partial \xi^2}{\partial x^1} + w_0 \frac{\partial \xi^3}{\partial x^1} = 0, \qquad (1.21)$$

$$\frac{\partial \eta^4}{\partial x^2} + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} +$$

$$+ u_0 \frac{\partial \xi^1}{\partial x^2} + v_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) + w_0 \frac{\partial \xi^3}{\partial x^2} = 0, \qquad (1.22)$$

$$\frac{\partial \eta^4}{\partial x^3} + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} +$$

$$+ u_0 \frac{\partial \xi^1}{\partial x^3} + v_0 \frac{\partial \xi^2}{\partial x^3} + w_0 \left(\frac{\partial \eta^2}{\partial u^2} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \eta^3}{\partial u^3} - \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \eta^1}{\partial u^1} + \frac{\partial \xi^4}{\partial x^4} \right) = 0. \qquad (1.23)$$

As in the previous cases, let us note that equations (1.21)–(1.23) differ from equations (1.15)–(1.17) by the terms in parentheses.

Therefore, the determining equations for the coordinates of the operator X that are admitted by system (0.5)–(0.8) are

$$\frac{\partial \eta^1}{\partial u^1} = \frac{\partial \eta^2}{\partial u^2} = \frac{\partial \eta^3}{\partial u^3}, \qquad (1.24)$$

$$\frac{\partial \eta^1}{\partial u^1} + \frac{\partial \eta^4}{\partial u^4} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} = 0, \qquad (1.25)$$

$$\frac{\partial \eta^4}{\partial x^1} + \xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} + u_0 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^2}{\partial x^2} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} \right) + v_0 \frac{\partial \xi^2}{\partial x^1} + w_0 \frac{\partial \xi^3}{\partial x^1} = 0, \ (1.26)$$

$$\frac{\partial \eta^4}{\partial x^2} + \xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} + u_0 \frac{\partial \xi^1}{\partial x^2} + v_0 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} \right) + w_0 \frac{\partial \xi^3}{\partial x^2} = 0, \ (1.27)$$

$$\frac{\partial \eta^4}{\partial x^3} + \xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} + u_0 \frac{\partial \xi^1}{\partial x^3} + v_0 \frac{\partial \xi^2}{\partial x^3} + w_0 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi^1}{\partial x^1} - \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \xi^4}{\partial x^4} \right) = 0. \ (1.28)$$

When $v_0 \neq 0$ and $w_0 \neq 0$ the following equations are valid

$$\frac{\partial \eta^2}{\partial u^1} + \frac{\partial \eta^1}{\partial u^2} = 0. \qquad (1.29)$$

$$\frac{\partial \eta^3}{\partial u^1} + \frac{\partial \eta^1}{\partial u^3} = 0, \qquad (1.30)$$

$$\frac{\partial \eta^3}{\partial u^2} + \frac{\partial \eta^2}{\partial u^3} = 0. \qquad (1.31)$$

According to equations (1.1), (1.2), (1.4), (1.12.) and (1.18) the following relation are true: $\eta^1(u^1, u^2, u^3)$, $\eta^2(u^1, u^2, u^3)$, $\eta^3(u^1, u^2, u^3)$, $\eta^4(x^1, x^2, x^3, u^4)$, $\xi^j(x^1, x^2, x^3)$, $j = 1, 2, 3$ and $\xi^4(x^4)$.

The analysis of the equations which do not contain initial velocity gives the preliminary structure of the sought-for functions:

$$\xi^i = \xi^i(x^1, x^2, x^3), \ i = 1, 2, 3,$$

$$\xi^4 = \xi^4(x^4),$$

$$\eta^1 = C_1 u^1 - C_2 u^2 - C_3 u^3 + C_4,$$

$$\eta^2 = C_2 u^1 + C_1 u^2 + C_5 u^3 + C_6,$$

$$\eta^3 = C_3 u^1 - C_5 u^2 + C_1 u^3 + C_7,$$

$$\eta^4 = \left(2C_1 - \xi^4_{x^4} \right) u^4 + \Phi(x^1, x^2, x^3, x^4),$$

here $C_1 \ldots C_7$ are constants. Then we obtain the equation that links ξ^1, ξ^2 and ξ^3:

$$\xi^1_{x^1} + \xi^2_{x^2} + \xi^3_{x^3} = 3C_1. \qquad (1.32)$$

The remaining three determining equations are in fact classifying equations for the functions $u_0(\boldsymbol{\xi})$, $v_0(\boldsymbol{\xi})$, $w_0(\boldsymbol{\xi})$:

$$\xi^1 \frac{\partial u_0}{\partial x^1} + \xi^2 \frac{\partial u_0}{\partial x^2} + \xi^3 \frac{\partial u_0}{\partial x^3} + u_0 \left(-2C_1 + \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^4}{\partial x^4} \right) + v_0 \frac{\partial \xi^2}{\partial x^1} + w_0 \frac{\partial \xi^3}{\partial x^1} + \frac{\partial \Phi}{\partial x^1} = 0, \quad (1.33)$$

$$\xi^1 \frac{\partial v_0}{\partial x^1} + \xi^2 \frac{\partial v_0}{\partial x^2} + \xi^3 \frac{\partial v_0}{\partial x^3} + u_0 \frac{\partial \xi^1}{\partial x^2} + v_0 \left(-2C_1 + \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \xi^4}{\partial x^4} \right) + w_0 \frac{\partial \xi^3}{\partial x^2} + \frac{\partial \Phi}{\partial x^2} = 0, \quad (1.34)$$

$$\xi^1 \frac{\partial w_0}{\partial x^1} + \xi^2 \frac{\partial w_0}{\partial x^2} + \xi^3 \frac{\partial w_0}{\partial x^3} + u_0 \frac{\partial \xi^1}{\partial x^3} + v_0 \frac{\partial \xi^2}{\partial x^3} + w_0 \left(-2C_1 + \frac{\partial \xi^3}{\partial x^3} + \frac{\partial \xi^4}{\partial x^4} \right) + \frac{\partial \Phi}{\partial x^3} = 0. \quad (1.35).$$

Let $u_0(\boldsymbol{\xi})$, $v_0(\boldsymbol{\xi})$ and $w_0(\boldsymbol{\xi})$ be arbitrary functions then $\xi^1 = 0$, $\xi^2 = 0$ and $\xi^3 = 0$. Therefore, we have $C_1 = 0$, $\partial \xi^4 / \partial x^4 = 0$, $\partial \Phi / \partial x^1 = 0$, $\partial \Phi / \partial x^2 = 0$ and $\partial \Phi / \partial x^3 = 0$. It means that $\xi^4 = C_9 = const$ and $\Phi = h(x^4)$. Then the basis of the main operators of the Lie algebra consists of the following operators

$$L_0: \ X_1 = \frac{\partial}{\partial x^4}, \ X_2 = -u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}, \ X_3 = -u^3 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^3}, \ X_4 = -u^2 \frac{\partial}{\partial u^3} + u^3 \frac{\partial}{\partial u^2},$$

$$X_5 = \frac{\partial}{\partial u^1}, \ X_6 = \frac{\partial}{\partial u^2}, \ X_7 = \frac{\partial}{\partial u^3}, \ X_8 = h(x^4) \frac{\partial}{\partial u^4}.$$

It should be noted that equations (1.33)–(1.35) do not contain terms which depend on x^4, with the exception of the term $\partial \xi^4 / (\partial x^4)$. Then we can conclude that $\partial \xi^4 / (\partial x^4) = C_8 = const$ so $\xi^4 = C_8 x^4 + C_9$.

It turns out that the analysis of system (1.33)–(1.35) is reduced to the analysis of three equations for the components of the initial vortex $\boldsymbol{\omega} = rot\mathbf{u_0}$, where $\mathbf{u_0} = (u_0, v_0, w_0)$, $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$, $\omega^1 = w_{0x^2} - v_{0x^3}$, $\omega^2 = u_{0x^3} - w_{0x^1}$ and $\omega^3 = v_{0x^1} - u_{0x^2}$. Indeed, considering the condition of compatibility of equations (1.19)–(1.21) we get the system of equations that contains only components of vector $\boldsymbol{\omega}$ and the operator coordinates ξ^1, ξ^2 and ξ^3

$$\left(C_8 - 2C_1 + \frac{\partial \xi^2}{\partial x^2} + \frac{\partial \xi^3}{\partial x^3} \right) \omega^1 - \frac{\partial \xi^1}{\partial x^2} \omega^2 - \frac{\partial \xi^1}{\partial x^3} \omega^3 + \xi^1 \omega_{x^1}^1 + \xi^2 \omega_{x^2}^1 + \xi^3 \omega_{x^3}^1 = 0, \quad (1.36)$$

$$\left(C_8 - 2C_1 + \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^3}{\partial x^3} \right) \omega^2 - \frac{\partial \xi^2}{\partial x^1} \omega^1 - \frac{\partial \xi^2}{\partial x^3} \omega^3 + \xi^1 \omega_{x^1}^2 + \xi^2 \omega_{x^2}^2 + \xi^3 \omega_{x^3}^2 = 0, \quad (1.37)$$

$$\left(C_8 - 2C_1 + \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^2}{\partial x^2} \right) \omega^3 - \frac{\partial \xi^3}{\partial x^1} \omega^1 - \frac{\partial \xi^3}{\partial x^2} \omega^2 + \xi^1 \omega_{x^1}^3 + \xi^2 \omega_{x^2}^3 + \xi^3 \omega_{x^3}^3 = 0. \quad (1.38)$$

Equations (1.36)–(1.38) supplemented by equation (1.32) are classifying equations for system (0.1)–(0.4). With the change of variables $\xi^1 = C_1 x^1 + \overline{\xi}^1$, $\xi^2 = C_1 x^2 + \overline{\xi}^2$, $\xi^3 = C_1 x^3 + \overline{\xi}^3$ equation (1.32) becomes homogeneous one

$$\overline{\xi}_{x^1}^1 + \overline{\xi}_{x^2}^2 + \overline{\xi}_{x^3}^3 = 0. \quad (1.39).$$

As a result of this change of variables, equations (1.36)–(1.38) take the form

$$\left(C_8 - \frac{\partial \overline{\xi}^1}{\partial x^1} \right) \omega^1 - \frac{\partial \overline{\xi}^1}{\partial x^2} \omega^2 - \frac{\partial \overline{\xi}^1}{\partial x^3} \omega^3 + (C_1 x^1 + \overline{\xi}^1) \omega_{x^1}^1 + (C_1 x^2 + \overline{\xi}^2) \omega_{x^2}^1 + (C_1 x^3 + \overline{\xi}^3) \omega_{x^3}^1 = 0, \ (1.40)$$

$$\left(C_8 - \frac{\partial \overline{\xi}^2}{\partial x^2} \right) \omega^2 - \frac{\partial \overline{\xi}^2}{\partial x^1} \omega^1 - \frac{\partial \overline{\xi}^2}{\partial x^3} \omega^3 + (C_1 x^1 + \overline{\xi}^1) \omega_{x^1}^2 + (C_1 x^2 + \overline{\xi}^2) \omega_{x^2}^2 + (C_1 x^3 + \overline{\xi}^3) \omega_{x^3}^3 = 0, \ (1.41)$$

$$\left(C_8 - \frac{\partial \bar{\xi}^3}{\partial x^3}\right)\omega^3 - \frac{\partial \bar{\xi}^3}{\partial x^1}\omega^1 - \frac{\partial \bar{\xi}^3}{\partial x^2}\omega^2 + (C_1 x^1 + \bar{\xi}^1)\omega_{x^1}^3 + (C_1 x^2 + \bar{\xi}^2)\omega_{x^2}^3 + (C_1 x^3 + \bar{\xi}^3)\omega_{x^3}^3 = 0. \quad (1.42)$$

We need to add equation

$$\omega_{x^1}^1 + \omega_{x^2}^2 + \omega_{x^3}^3 = 0. \quad (1.43)$$

to equations (1.39)–(1.42).

Therefore, we have five equations (1.39)–(1.43) for six functions $\bar{\xi}^j$, ω^j, $j = 1, 2, 3$.

2.   The solution of classifying equations

From equations (1.40)–(1.42) the first classifying relation $\omega^i = const$ can be derived. Then system of equations (1.40)–(1.42) can be rewritten as

$$\frac{\partial \bar{\xi}^1}{\partial x^1}\omega^1 + \frac{\partial \bar{\xi}^1}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^1}{\partial x^3}\omega^3 = C_8 \omega^1,$$

$$\frac{\partial \bar{\xi}^2}{\partial x^1}\omega^1 + \frac{\partial \bar{\xi}^2}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^2}{\partial x^3}\omega^3 = C_8 \omega^2,$$

$$\frac{\partial \bar{\xi}^3}{\partial x^1}\omega^1 + \frac{\partial \bar{\xi}^3}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^3}{\partial x^3}\omega^3 = C_8 \omega^3.$$

Thus, from equations (1.40)–(1.42) we obtain the following equations

$$\triangledown \bar{\xi}^j \cdot \boldsymbol{\omega} = C_8 \omega^j, \quad (2.1)$$

$j = 1, 2, 3$, $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$ is constant vorticity of the fluid at the initial point in time. Let us assume that $\boldsymbol{\omega} \neq 0$ and without loss of generality we can also assume that $\omega^1 = const \neq 0$. Then equation (2.1) can be divided by ω^1 and vorticity takes the form $\boldsymbol{\omega} = (1, \omega^2, \omega^3)$ with new constants ω^2 and ω^3.

Let us rewrite system of equations (2.1) in extended form, assuming that $\omega^1 = 1$,

$$\frac{\partial \bar{\xi}^1}{\partial x^1} + \frac{\partial \bar{\xi}^1}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^1}{\partial x^3}\omega^3 = C_8, \quad (2.2)$$

$$\frac{\partial \bar{\xi}^2}{\partial x^1} + \frac{\partial \bar{\xi}^2}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^2}{\partial x^3}\omega^3 = C_8 \omega^2, \quad (2.3)$$

$$\frac{\partial \bar{\xi}^3}{\partial x^1} + \frac{\partial \bar{\xi}^3}{\partial x^2}\omega^2 + \frac{\partial \bar{\xi}^3}{\partial x^3}\omega^3 = C_8 \omega^3. \quad (2.4)$$

We obtain three first order partial differential equations in variables $\bar{\xi}^1$, $\bar{\xi}^2$ and $\bar{\xi}^3$.

One can suggest the following solution of equations (2.2)–(2.4)

$$\bar{\xi}^1 = C_8 x^1 + f^1(\alpha, \beta), \quad (2.5)$$

$$\bar{\xi}^2 = C_8 \omega^2 x^1 + f^2(\alpha, \beta), \quad (2.6)$$

$$\bar{\xi}^3 = C_8 \omega^3 x^1 + f^3(\alpha, \beta), \quad (2.7)$$

where $\alpha = x^2 - \omega^2 x^1$, $\beta = x^3 - \omega^3 x^1$.

If we assume $f^2(\alpha, \beta) = \widetilde{f}^2(\alpha, \beta) + \alpha C_8$ then instead of $C_8 \omega^2 x^1$ we can write $C_8 x^2$ in equation (2.6). Similarly, instead of $C_8 \omega^3 x^1$ we can write $C_8 x^3$ in equation (2.6).

Remark. When $\omega^2 \neq 0$, $\omega^3 = 0$; $\omega^2 = 0$, $\omega^3 \neq 0$; $\omega^2 = \omega^3 = 0$ we have special cases of relations (2.5)–(2.7):

a) $\omega^2 \neq 0$, $\omega^3 = 0$: $\bar{\xi}^1 = C_8 x^1 + f^1(\alpha, x^3)$, $\bar{\xi}^2 = C_8 x^2 + f^2(\alpha, x^3)$, $\bar{\xi}^3 = f^3(\alpha, x^3)$.

b)$\omega^2 = 0$, $\omega^3 \neq 0$: $\bar{\xi}^1 = C_8 x^1 + f^1(x^2, \beta)$, $\bar{\xi}^2 = f^2(x^2, \beta)$, $\bar{\xi}^3 = C_8 x^3 + f^3(x^2, \beta)$

c)$\omega^2 = \omega^3 = 0$ $\bar{\xi}^1 = C_8 x^1 + f^1(x^2, x^3)$, $\bar{\xi}^2 = f^2(x^2, x^3)$, $\bar{\xi}^3 = f^3(x^2, x^3)$.

After substituting expressions (2.5)–(2.7) into equation (1.39), we obtain the following equation

$$3C_8 + (f^2 - \omega^2 f^1)_\alpha + (f^3 - \omega^3 f^1)_\beta = 0. \tag{2.8}$$

Let us introduce the following designations in equation (2.8): $d_\beta = f^2 - \omega^2 f^1$ and $d_\alpha = f^3 - \omega^3 f^1 + 3C_8\beta$. Then we find that $f^2 = \omega^2 f^1 + d_\beta$, $f^3 = \omega^3 f^1 - 3C_8\beta + d_\alpha$.

Conclusion

For the vorticity vector $\boldsymbol{\omega} = (1, \omega^2, \omega^3) = const$ we have the following coordinates of the operator

$$\xi^1 = C_1 x^1 + C_8 x^1 + f^1(\alpha, \beta),$$

$$\xi^2 = C_1 x^2 + C_8 x^2 + \omega^2 f^1 + d_\beta,$$

$$\xi^3 = C_1 x^3 + C_8 x^3 + \omega^3 f^1 - 3C_8\beta + d_\alpha,$$

$$\xi^4 = C_8 x^4 + C_9,$$

$$\eta^1 = C_1 u^1 - C_2 u^2 - C_3 u^3 + C_5,$$

$$\eta^2 = C_2 u^1 + C_1 u^2 + C_4 u^3 + C_6,$$

$$\eta^3 = C_3 u^1 - C_4 u^2 + C_1 u^3 + C_7,$$

$$\eta^4 = (2C_1 - C_8) u^4 + h(x^1, x^2, x^3) + \varphi(x^4),$$

where $\alpha = x^2 - \omega^2 x^1$, $\beta = x^3 - \omega^3 x^1$ and functions $h(x^1, x^2, x^3)$, $\varphi(x^4)$, $f^1(\alpha, \beta)$, $d(\alpha, \beta)$ are arbitrary functions.

The basic Lie algebra L_0 is extended by operators

$$X_1 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + u^1 \frac{\partial}{\partial u^1} + u^3 \frac{\partial}{\partial u^3} + u^3 \frac{\partial}{\partial u^3} + 2u^4 \frac{\partial}{\partial u^4},$$

$$X_2 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4} - u^4 \frac{\partial}{\partial u^4},$$

$$X_3 = \frac{\partial}{\partial x^1} + \omega^2 \frac{\partial}{\partial x^2} + \omega^3 \frac{\partial}{\partial x^3}, \ X_4 = d_\beta \frac{\partial}{\partial x^2} - d_\alpha \frac{\partial}{\partial x^3}, \ X_5 = h(x^1, x^2, x^3) \frac{\partial}{\partial u^4}.$$

Another possibility to obtain classifying equations is given by the function $\boldsymbol{\omega}$. Let us present system of equations (1.40)–(1.42) in the form

$$C_1 \mathbf{x} \cdot \bigtriangledown \omega^k + \boldsymbol{\xi} \cdot \bigtriangledown \omega^k + C_8 \omega^k - \bigtriangledown \xi^k \cdot \boldsymbol{\omega} = 0, \tag{2.9}$$

$k = 1, 2, 3$, C_1 and C_8 are some constants,

$$div\boldsymbol{\xi} = 0, \ div\boldsymbol{\omega} = 0. \tag{2.10}$$

Equations (2.9) and (2.10) allow one to determine the coordinates of the operator ξ in terms of ω.

The basis of the Lie algebra for any function $\boldsymbol{\omega}$ that extends L_0 can be found in the following cases:

1) if $C_1 = 1$, $C_8 = 0$;

2) if $C_1 = 0$, $C_8 = 1$;

3) if $C_1 = 0$, $C_8 = 0$.

In the first case the coordinates of the operator are derived from system (2.10) and equations

$$\mathbf{x} \cdot \bigtriangledown \omega^k + \boldsymbol{\xi} \cdot \bigtriangledown \omega^k - \bigtriangledown \xi^k \cdot \boldsymbol{\omega} = 0,$$

$k = 1,\ 2,\ 3$. In the second case the coordinates of the operator are derived from system (2.10) and equations

$$\boldsymbol{\xi} \cdot \bigtriangledown \omega^k + \omega^k - \bigtriangledown \xi^k \cdot \boldsymbol{\omega} = 0,$$

$k = 1,\ 2,\ 3$. In the third case the coordinates of the operator are derived from system (2.10) and equations

$$\boldsymbol{\xi} \cdot \bigtriangledown \omega^k - \bigtriangledown \xi^k \cdot \boldsymbol{\omega} = 0,$$

$k = 1,\ 2,\ 3$. A special solution of the latter system of equations is $\boldsymbol{\xi} = \boldsymbol{\omega}$. For this solution we have the following coordinates of the operator

$$\xi^i = \omega^i(x^1,\ x^2,\ x^3),\ i = 1,\ 2,\ 3,\ \ \xi^4 = \xi^4(x^4),$$

$$\eta^1 = -C_2 u^2 - C_3 u^3 + C_4,\ \ \eta^2 = C_2 u^1 + +C_5 u^3 + C_6,$$

$$\eta^3 = C_3 u^1 - C_5 u^2 + +C_7,\ \ \eta^4 = -\xi^4_{x^4} u^4 + \Phi(x^1, x^2, x^3, x^4).$$

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Групповая классификация уравнений трёхмерной идеальной жидкости в терминах траекторий и потенциала Вебера

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Проводится групповой анализ уравнений движения идеальной жидкости в переменных траекторий — потенциал Вебера. Показано, что переход к произвольным лагранжевым координатам, сохраняющий объём, является преобразованием эквивалентности для этой системы. Получены классифицирующие уравнения на функции начального распределения скорости. Вычислена основная группа Ли и указаны её расширения.

Ключевые слова: уравнения идеальной жидкости, преобразование эквивалентности, лагранжевы координаты, классифицирующие уравнения.