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The Linearity Problem for the Unitriangular Automorphism Groups of Free Groups

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We prove that the unitriangular automorphism group of a free group of rank n has a faithful representation by matrices over a field, or in other words, it is a linear group, if and only if $n \leq 3$. Thus, we have completed a description of relatively free groups with linear the unitriangular automorphism groups. This description was initiated by Erofeev and the author in [1], where proper varieties of groups have been considered.

Keywords: free group, unitriangular automorphism, linearity.

Introduction

For each positive integer n, let F_n be a free group of rank n with basis (in other words, free generating set) $\{f_1, ..., f_n\}$. For any $m \leqslant n$ F_m is considered as subgroup $\mathrm{gp}(f_1, ..., f_m)$ of F_n. For any variety of groups $\mathcal{G}$, let $\mathcal{G}(F_n)$ denote the verbal subgroup of F_n corresponding to $\mathcal{G}$. Let $G_n = F_n/\mathcal{G}(F_n)$. Then G_n is a relatively free group of rank n in the variety $\mathcal{G}$. By basis of G_n we mean a subset S such that every map of S into G_n extends, uniquely, to an endomorphism of G_n. Write $\bar{f}_i = f_i\mathcal{G}(F_n)$ for $i = 1, ..., n$. Then $\bar{f}_1, ..., \bar{f}_n$ is a basis of G_n.

Let $\mathcal{G}$ be a variety of groups. Let G_n be the relatively free group corresponding to $\mathcal{G}$ with basis $\{f_1, ..., f_n\}$. For any $m \leqslant n$ G_m is considered as subgroup $\mathrm{gp}(f_1, ..., f_m)$ of G_n. An automorphism φ of G_n is called *unitriangular* (w.r.t. the given basis) if φ is defined by a map of the form:

$$\varphi : f_1 \mapsto f_1, f_i \mapsto u_i f_i \text{ для } i = 2, ..., n, \tag{1}$$

where $u_i = u_i(f_1, ..., f_{i-1})$ is an element of G_{i-1}. Every tuple of elements $(u_2, ..., u_n)$ with this condition defines, uniquely, automorphism of G_n. Let U_n be subgroup consisting of all unitriangular (w.r.t. a given basis) automorphisms of G_n. Then it is called *the unitriangular automorphism group* of G_n. As abstract group U_n does not depend of a basis.

The question of linearity of U_n for an arbitrary proper variety $\mathcal{G}$ has been studied by Erofeev and the author in [1]. All cases of linearity of G_n have been described. The following Section 1 contains this description. Also, we observe some relative results on linearity for relatively free groups and algebras.

In this paper we study the only open after [1] case when $\mathcal{G}$ is the variety of all groups. Our main result is given by the following theorem.

Theorem 1. *The group U_n of unitriangular automorphisms of the free group F_n of rank n is linear if and only if $n \leqslant 3$.*

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Hence, we complete a description of all cases when the unitriangular automorphism group U_n, corresponding to an arbitrary variety of groups $\mathcal{G}$, including the variety of all groups, is linear.

1. Some results on linearity

We observe some results concerning the linearity of the automorphism groups and their subgroups of relatively free groups and algebras. Recall that group G is said to be *virtually nilpotent* if it has a nilpotent subgroup of finite index.

The linearity of $\mathrm{Aut}(F_2)$ follows by [2] from the linearity of the 4−string braid group B_4, which is due to Krammer [3]. Bigelow [4] and also Krammer [5] determined that the braid group B_n is linear for every n. Formanek and Procesi in [6] have demonstrated that $\mathrm{Aut}(F_n)$ is not linear for $n \geqslant 3$.

Auslander and Baumslag [7] determined that for every finitely generated virtually nilpotent group G the automorphism group $\mathrm{Aut}(G)$ is linear. Moreover, $\mathrm{Aut}(G)$ has a faithful matrix representation over the integers $\mathbb{Z}$. In particular, for every relatively free virtually nilpotent group G_n, the automorphism group $\mathrm{Aut}(G_n)$ is linear over $\mathbb{Z}$.

Olshanskii [8] proved for any relatively free group G_n, which is not virtually nilpotent and is not free, that the automorphism group $\mathrm{Aut}(G_n)$ is not linear. His approach does not give an information on the linearity of the unitriangular automorphism groups U_n for such relatively free groups G_n.

Erofeev and the author [1] proved for every proper variety of groups $\mathcal{G}$ that the unitriangular automorphism group U_n is linear if and only if the relatively free group G_{n-1} is virtually nilpotent. More exactly (for $n \geqslant 3$): if G_{n-1} is virtually nilpotent, then U_n admits a faithful matrix representation over integers $\mathbb{Z}$. It was also shown in [1] that if $n \geqslant 3$ and G_{n-1} is nilpotent then U_n is nilpotent too.

Now let C_n be an arbitrary relatively free algebra of rank n with set of free generating elements $\{x_1, ..., x_n\}$. For $m \leqslant n$ C_m can be considered as subalgebra of C_n generated by $x_1, ..., x_m$. An automorphism ψ of C_n is called *unitriangular* w.r.t. the given set of free generating elements if it is defined by map of the form:

$$\psi : x_1 \mapsto x_1, x_i \mapsto x_i + u_i \text{ для } i = 2, ..., n, \tag{2}$$

where $u_i = u_i(x_1, ..., x_{i-1})$ belongs to C_{i-1}. Let U_n denote a subgroup of the automorphism group $\mathrm{Aut}(C_n)$ of C_n, consisting of all unitriangular automorphisms. As abstract group U_n does not depend from a chosen set of free generating elements of C_n.

The author, Chirkov and Shevelin [9] proved that, for a free Lie (free associative, absolutely free, polynomial) algebra C_n of rank $n \geqslant 4$ over a field of zero characteristic, the unitriangular automorphism group U_n is not linear. Then the following papers [10, 11] presented descriptions of the hypercentral series of groups U_n corresponding to polynomial and free metabelian Lie algebras, respectively. By these results U_n are not linear for $n \geqslant 3$. By [12], for $n \geqslant 3$, the unitriangular automorphism group U_n is not linear in case of polynomial algebra and in case of free associative algebra. By [13] for each relatively free algebra C_n the group U_n is locally nilpotent, thus it is linear.

2. The method of Formanek and Procesi

Let G be any group, and let $\mathcal{H}(G)$ denote the following HNN-extension of $G \times G$:

$$\mathcal{H}(G) =< G \times G, t : t(g, g)t^{-1} = (1, g), g \in G > . \tag{3}$$

Theorem 2.1 (Formanek, Procesi [6]). *Let ρ be a linear representation of $\mathcal{H}(G)$. Then the image of $G \times \{1\}$ has a subgroup of finite index with nilpotent derived subgroup, i.e, is nilpotent-by-abelian-by-finite.*

Theorem 2.2 (Brendle, Hamidi-Tehrani [14]). *Let N be a normal subgroup of $\mathcal{H}(G)$ such that the image of $G \times \{1\}$ in $\mathcal{H}(G)/N$ is not nilpotent-by-abelian-by-finite. Then $\mathcal{H}(G)/N$ is not linear.*

In [14] a group of the type described in Theorem 2.2 is called a *Formanek and Procesi group*, or *FP-group* for short.

3. Proof of Theorem 1

2.1. For $n \leqslant 3$, U_n is linear.

Proof. Since U_1 is trivial and U_2 is infinite cyclic the statement is obvious for $n = 1, 2$.

Let $n = 3$. By [1] U_3 is generated by automorphisms $\lambda_{2,1}, \lambda_{3,1}, \lambda_{3,2}$. Recall that $\lambda_{i,j}$ maps f_i to $f_j f_i$, and fixes all other basic elements. This is applicable for any group U_n. The automorphisms $\lambda_{3,1}$ $\lambda_{3,2}$ generate in U_3 a normal free subgroup F_2. The automorphism $\lambda_{2,1}$ acts as follows:

$$\lambda_{2,1}^{-1}\lambda_{3,1}\lambda_{2,1} = \lambda_{3,1},\ \lambda_{2,1}^{-1}\lambda_{3,2}\lambda_{2,1} = \lambda_{3,1}\lambda_{3,2}. \tag{4}$$

Now we'll show that U_3 is isomorphic to a subgroup of $\mathrm{Aut}(F_2)$. Let τ_1 and τ_2 denote inner automorphisms of F_2 corresponding to f_1 and f_2 respectively. This means that any element g of F_2 maps by $\tau_i (i = 1, 2)$ to $f_i^{-1} g f_i$. Let $\sigma_{2,1} \in \mathrm{Aut}(F_2)$ fixes f_1 and maps f_2 to $f_1 f_2$. Obviously, $F_2 = \mathrm{gp}(\tau_1, \tau_2)$ is a free group of rank 2. It is a normal subgroup of $V_3 = \mathrm{gp}(\tau_1, \tau_2, \sigma_{2,1})$. A quotient V_3/F_2 is the infinite cyclic generated by the image of $\sigma_{2,1}$. The corresponding action is determined by:

$$\sigma_{2,1}^{-1}\tau_1\sigma_{2,1} = \tau_1,\ \sigma_{2,1}^{-1}\tau_2\sigma_{2,1} = \tau_1\tau_2. \tag{5}$$

Thus, U_3 and V_3 are both infinite cyclic extensions of F_2. By (4) and (5) we conclude that $\alpha : U_3 \to V_3$ defined as:

$$\alpha : \lambda_{3,j} \mapsto \tau_j \text{ для } j = 1, 2,\ \lambda_{2,1} \mapsto \sigma_{2,1}, \tag{6}$$

is isomorphism. Since V_3 is a subgroup of $\mathrm{Aut}(F_2)$, which is linear by [2] and [3], U_3 is also linear. □

2.2. For $n \geqslant 4$, U_n is not linear.

Proof. For $n \geqslant m$, U_n has a subgroup that is isomorphic to U_m. Elements of this subgroup act naturally to $f_1, ..., f_m$ and fix elements $f_{m+1}, ..., f_n$. So, we just have to prove that U_4 is not linear.

By Theorem 2.2 it will be enough to find a subgroup H of U_4 that is isomorphic to a quotient $\mathcal{H}(F_2)/N$, where $\mathcal{H}(F_2)$ is given by (3), such that the image of $G \times \{1\}$ in $\mathcal{H}(G)/N$ is not nilpotent-by-abelian-by-finite.

There are two commuting elementwise subgroups of U_4 each of them is isomorphic to F_2. Namely, there are $\mathrm{gp}(\lambda_{3,1}, \lambda_{3,2})$ and $\mathrm{gp}(\lambda_{4,1}, \lambda_{4,2})$. Consider them as two copies of F_2 via isomorphism defined by map $\lambda_{3,1} \mapsto \lambda_{4,1}, \lambda_{3,2} \mapsto \lambda_{4,2}$. Thus we have a subgroup $F_2 \times F_2$ of U_4. Easily to check that:

$$\lambda_{4,3}^{-1}\lambda_{3,j}\lambda_{4,j}\lambda_{4,3} = \lambda_{3,j},\ j = 1, 2. \tag{7}$$

By (3) and (7) we conclude that H is a homomorphic image of $\mathcal{H}(F_2)$ such that the subgroup $F_2 \times F_2$ of $\mathcal{H}(F_2)$ maps isomorphically to the just constructed subgroup of the same type of U_4. The image of t is $\lambda_{4,3}$. Hence, $H \simeq \mathcal{H}(F_2)/N$, where N is the kernel of this homomorphism. By Theorem 2.2 H, and so U_4, is not linear. □

Remark 1. *In fact we proved that subgroup* $W_4 = gp(\lambda_{3,j}, \lambda_{4,l}\ : j = 1,2; l = 1,2,3)$ *of* U_4 *is not linear. In [1] we noted that* $[\lambda_{i,j}, \lambda_{j,k}] = \lambda_{i,k}$. *Here commutator* $[g, f]$ *means* $gfg^{-1}f^{-1}$. *Any group* U_n *is generated by the elements* $\lambda_{i,j}$, *for* $j < i \leqslant n$ *(see [1]). It follows that* W_4 *is the derived subgroup* U_4' *of* U_4. *Hence, we proved that the derived subgroup (the second member* $\gamma_2 U_4$ *of the low central series) of* U_4 *is not linear. This subgroup* $\gamma_2 U_4$ *has also characterized in* U_4 *as the stabilizer of* f_1. *In general case, for* $n \geqslant 4$, *member* $\gamma_{n-2}U_n$ *coincides with the elementwise stabilizer of* $\{f_1, ..., f_{n-3}\}$. *Easily to see that the derived subgroup* U_4' *can be embedded into* $\gamma_{n-2}U_n$. *Hence, for every* $n \geqslant 4$, *a member* $\gamma_{n-2}U_n$ *of the low central series of* U_n *is not linear. This statement is more strong than the statement of Theorem 1 about nonlinearity of* U_n *for* $n \geqslant 4$.

Remark 2. *In [15] an explicit faithful representation of* $Aut(F_2)$ *in* $GL_{12}(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$ *is given. Here* $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ *is a Laurent polynomial ring. Hence,* U_3 *has a faithful matrix representation over* $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

Also note that U_3 *can be presented by* $< a, b : [[a, b], b]] = 1 >$, *where* a *corresponds to* $\lambda_{3,2}$, *and* b *corresponds to* $\lambda_{2,1}$. *By terminology of [17]* U_3 *is the first non commutative member in a series of Hydra groups* $H_k =< a, b : [...[[a, b], b], ..., b] = 1 >, k \geqslant 1$, *where commutator has* k *entries of* b. *In general, Hydra groups were introduced in [16]. It was shown in [17] that Hydra groups of such form are residually torsion-free nilpotent. It seems interesting to study their linearity.*

Remark 3. *By [18] a group* G *is called locally graded if every nontrivial finitely generated subgroup of* G *has a proper subgroup of finite index. This class contains, for example, all locally solvable and all residually finite groups. Let* G_2 *be the relatively free group of rank* 2 *in the variety* $var(G)$ *generated by* G. *Suppose that the derived subgroup* G_2' *is finitely generated. Then by [18]* G *is virtually nilpotent.*

Let $\mathcal{G}$ be a variety consisting of locally graded groups. Obviously $\mathcal{G}$ is a proper variety. Suppose that U_3 is linear. Then every group U_n is linear. Indeed, by [1] G_2 is virtually nilpotent. It follows that G_2' is finitely generated. Any group G_{n-1} generates a subvariety $\mathrm{var}(G_{n-1})$ of $\mathcal{G}$. The relatively free group of rank 2 in this subvariety is a homomorphic image of G_2, and so has finitely generated derived subgroup. Then by [18] G_{n-1} is virtually nilpotent. It follows by [1] that U_n is linear. Thus, the linearity of U_3 implies the linearity of U_n for every $n \geqslant 4$. Moreover, $\mathcal{G}$ should be virtually nilpotent.

We see by Theorem 1 that just presented statement, that the linearity of U_3 implies the linearity of U_n for all $n \geqslant 4$, is not true for the variety of all groups. Likely, it is also non-true for some proper varieties of groups. As candidates to such varieties we can consider the varieties of groups generated by the famous Golod groups. We conjecture that for every $m \geqslant 3$ there is a variety $\mathcal{G}_m$ such that the groups U_n are linear if and only if $n \leqslant m$.

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Проблема линейности групп унитреугольных автоморфизмов свободных групп

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Доказано, что группа унитреугольных автоморфизмов свободной группы ранга n допускает точное представление матрицами над полем тогда и только тогда, когда $n \leqslant 3$. Таким образом завершено описание относительно свободных групп, группы унитреугольных автоморфизмов которых линейны, начатое работой С. Ю. Ерофеева и автора [1] , где рассмотрены все относительно свободные группы собственных многообразий групп.

Ключевые слова: свободная группа, унитреугольный автоморфизм, линейность.