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On Solvability of an Inverse Boundary Value Problem for the Boussinesq–Love Equation

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In the paper an inverse boundary value problem for the Boussinesq-Love equation with an integral condition of the first kind is investigated. First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. The existence and uniqueness of a solution to the system of integral equation is proved by the contraction mapping principle. This solution is also the unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved.

Keywords: inverse boundary value problem, the Boussinesq–Love equation, Fourier method, classical solution.

Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics.

Inverse problems for various types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2, 3], V.K. Ivanov [4] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [5].

In this paper, following [6–9], we prove existence and uniqueness of the solution to an inverse boundary value problem for the Boussinesq-Love equation modeling the longitudinal waves in an elastic bar with the transverse inertia.

1. Problem statement and its reduction to an equivalent problem

Consider for the Boussinesq-Love equation [10]

$$u_{tt}(x,t) - u_{ttxx}(x,t) - \alpha u_{txx}(x,t) - \beta u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T\}$ an inverse boundary problem with the initial conditions

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$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x) \quad (0\leqslant x\leqslant 1), \tag{2}$$

the periodic condition

$$u(0,t)=u(1,t) \quad (0\leqslant t\leqslant T), \tag{3}$$

the non-local integral condition

$$\int_0^1 u(x,t)dx=0 \quad (0\leqslant t\leqslant T) \tag{4}$$

and with the additional condition

$$u(x_0,t)=h(t) \quad (0\leqslant t\leqslant T), \tag{5}$$

where $x_0\in(0,1)$, $\alpha>0$, $\beta>0$ are the given numbers, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are the given functions, and $u(x,t)$, $a(t)$ are the required functions.

The condition (4) is a non-local integral condition of first kind, i.e. the one not involving values of unknown functions at the domain's boundary points.

Definition. *A classical solution to the problem* (1)–(5) *is a pair* $\{u(x,\,t),\,a(t)\}$ *of the functions* $u(x,t)$ *and* $a(t)$ *with the following properties*

1) the function $u(x,t)$ *is continuous in* D_T *together with all its derivatives contained in equation (1);*

2) the function $a(t)$ *is continuous on* $[0,T]$*;*

3) all the conditions (1)–(5) *are satisfied in the ordinary sense.*

The following lemma holds.

Lemma 1. *Let* $f(x,t)\in C(D_T)$, $\int_0^1 f(x,t)dx=0$ $(0\leqslant t\leqslant T)$, $\varphi(x),\psi(x)\in C^1[0,1]$, $h(t)\in C^2[0,T]$, $h(t)\neq 0$ $(0\leqslant t\leqslant T)$ *and*

$$\varphi'(0)=\varphi'(1),\ \psi'(0)=\psi'(1),$$
$$\int_0^1\varphi(x)dx=0,\ \int_0^1\psi(x)dx=0,\ \varphi(x_0)=h(0), \quad \psi(x_0)=h'(0).$$

Then the problem of finding a classical solution to the problem (1)–(5) *is equivalent to the problem of finding functions* $u(x,t)$ *and* $a(t)$ *with the properties 1) and 2) of the definition of the classical solution from the relations* (1)–(3) *and satisfying*

$$u_x(0,t)=u_x(1,t) \quad (0\leqslant t\leqslant T), \tag{6}$$

$$h''(t)-u_{ttxx}(x_0,t)-\alpha u_{txx}(x_0,t)-\beta u_{xx}(x_0,t)=a(t)h(t)+f(x_0,t) \quad (0\leqslant t\leqslant T). \tag{7}$$

Proof. Let $\{u\,(x,t)\,,a\,(t)\}$ be a classical solution to the problem (1)–(5). Integrating equation (1) with respect to x from 0 to 1, we have

$$\begin{gathered}\frac{d^2}{dt^2}\int_0^1 u(x,t)dx-\frac{d^2}{dt^2}\left(u_x(1,t)-u_x(0,t)\right)-\alpha\frac{d}{dt}\left(u_x(1,t)-u_x(0,t)\right)-\\ -\beta\left(u_x(1,t)-u_x(0,t)\right)=a(t)\int_0^1 u(x,t)dx+\int_0^1 f(x,t)dx \quad (0\leqslant t\leqslant T).\end{gathered} \tag{8}$$

Taking into account that $\int_0^1 f(x,t)dx=0$ $(0\leqslant t\leqslant T)$ and (4), we find that

$$\begin{gathered}-\frac{d^2}{dt^2}\left(u_x(1,t)-u_x(0,t)\right)-\alpha\frac{d}{dt}\left(u_x(1,t)-u_x(0,t)\right)-\\ -\beta\left(u_x(1,t)-u_x(0,t)\right)=0\ (0\leqslant t\leqslant T).\end{gathered} \tag{9}$$

By (2) and $\varphi'(0) = \varphi'(1),\ \psi'(0) = \psi'(1)$ we obtain

$$\begin{aligned} u_x(1,0) - u_x(0,0) &= \varphi'(1) - \varphi'(0) = 0, \\ u_{tx}(1,0) - u_{tx}(0,0) &= \psi'(1) - \psi'(0) = 0. \end{aligned} \tag{10}$$

Since the problem (9), (10) has only a trivial solution, we have $u_x(1,t) - u_x(0,t) = 0$, i.e. the condition (6) is fulfilled.

Assume now that $h(t) \in C^2[0,T]$. Differentiating (5) twice, we get

$$u_t(x_0,t) = h'(t), \quad u_{tt}(x_0,t) = h''(t) \quad (0 \leqslant t \leqslant T). \tag{11}$$

It follows from (1) that

$$u_{tt}(x_0,t) - u_{ttxx}(x_0,t) - \alpha u_{txx}(x_0,t) - \beta u_{xx}(x_0,t) = a(t)u(x_0,t) + f(x_0,t) \quad (0 \leqslant t \leqslant T). \tag{12}$$

Hence, taking into account (5) and (11), we conclude that (7) is fulfilled.

Now suppose that $\{u\,(x,t)\,, a\,(t)\}$ is a solution to the problem (1)–(3), (6), (7), then from (8) and (6) we find that

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - a(t)\int_0^1 u(x,t)dx = 0 \quad (0 \leqslant t \leqslant T). \tag{13}$$

By (2) and $\int_0^1 \varphi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0$, it is obvious that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \quad \int_0^1 u_t(x,0)dx = \int_0^1 \psi(x)dx = 0. \tag{14}$$

Since the problem (13), (14) has only a trivial solution, $\int_0^1 u(x,t)dx = 0 \ \ (0 \leqslant t \leqslant T)$, i.e. the condition (4) is fulfilled.

From (7) and (12) we obtain

$$\frac{d^2}{dt^2}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)) \quad (0 \leqslant t \leqslant T). \tag{15}$$

By (2) and $\varphi(x_0) = h(0),\ \psi(x_0) = h'(0)$ we have

$$\begin{cases} u(x_0,0) - h(0) = \varphi(x_0) - h(0) = 0, \\ u_t(x_0,0) - h'(0) = \psi(x_0) - h'(0) = 0. \end{cases} \tag{16}$$

From (15) and (16) we conclude that the condition (5) is fulfilled. The lemma is proved. □

2. Existence and uniqueness of the classical solution to the inverse boundary value problem

It is known [11] that the system

$$1, \cos\lambda_1 x,\ \sin\lambda_1 x, \ldots, \cos\lambda_k x,\ \sin\lambda_k x, \ldots \tag{17}$$

is a basis in $L_2(0,1)$, where $\lambda_k = 2k\pi \quad (k = 1,2,\ldots)$. Therefore, it is obvious that for each solution $\{u(x,t), a(t)\}$ to the problem (1)–(3), (6), (7) its first component $u(x,t)$ has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x \quad (\lambda_k = 2\pi k), \tag{18}$$

where

$$u_{10}(t)=\int_0^1 u(x,t)dx,$$
$$u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx,\quad u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx\quad (k=1,2,\ldots).$$

Then, applying the formal scheme of the Fourier method, from (1) и (2) we have

$$u''_{10}(t)=F_{10}(t;u,a)\quad (0\leqslant t\leqslant T), \tag{19}$$

$$\begin{gathered}\left(1+\lambda_k^2\right)u''_{ik}(t)+\alpha\lambda_k^2u'_{ik}(t)+\beta\lambda_k^2u_{ik}(t)=F_{ik}(t;u,a)\\ (0\leqslant t\leqslant T;\quad i=1,2;\quad k=1,2,\ldots),\end{gathered} \tag{20}$$

$$u_{10}(0)=\varphi_{10},\quad u'_{10}(0)=\psi_{10}, \tag{21}$$

$$u_{ik}(0)=\varphi_{ik},\quad u'_{ik}(0)=\psi_{ik}\quad (i=1,2;\quad k=1,2,\ldots), \tag{22}$$

where

$$F_{1k}(t;u,a)=a(t)u_{1k}(t)+f_{1k}(t)\quad (k=0,1,\ldots),$$
$$f_{10}(t)=\int_0^1 f(x,t)dx,\quad f_{1k}(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx\quad (k=1,2,\ldots),$$
$$\varphi_{10}=\int_0^1\varphi(x)dx,\quad \psi_{10}=\int_0^1\psi(x)dx,$$
$$\varphi_{1k}=2\int_0^1\varphi(x)cos\lambda_k x dx,\quad \psi_{1k}=2\int_0^1\psi(x)cos\lambda_k x dx\quad (k=1,2,\ldots),$$
$$F_{2k}(t;u,a)=a(t)u_{2k}(t)+f_{2k}(t),\quad f_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx\quad (k=1,2,\ldots),$$
$$\varphi_{2k}=2\int_0^1\varphi(x)\sin\lambda_k x dx,\quad \psi_{2k}=2\int_0^1\psi(x)\sin\lambda_k x dx\quad (k=1,2,\ldots).$$

It is obvious that $\lambda_k^2<1+\lambda_k^2<2\lambda_k^2$. Therefore

$$\frac{\alpha^2}{8}-\beta<\frac{\alpha^2\lambda_k^2}{4\left(1+\lambda_k^2\right)}-\beta<\frac{\alpha^2}{4}-\beta.$$

Now suppose that $\dfrac{\alpha^2}{8}-\beta>0$. Solving the problem (19)–(22), we find

$$u_{10}(t)=\varphi_{10}+t\psi_{10}+\int_0^t(t-\tau)F_{10}(\tau;u,a)d\tau\quad (0\leqslant t\leqslant T), \tag{23}$$

$$\begin{gathered}u_{ik}(t)=\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t}-\mu_{1k}e^{\mu_{2k}t}\right)\varphi_{ik}+\left(e^{\mu_{2k}t}-e^{\mu_{1k}t}\right)\psi_{ik}+\right.\\ \left.+\int_0^t F_{ik}(\tau;u,a)\left(e^{\mu_{2k}(t-\tau)}-e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\ (i=1,2;\quad 0\leqslant t\leqslant T),\end{gathered} \tag{24}$$

where

$$\begin{gathered}\mu_{ik}=-\frac{\alpha\lambda_k^2}{2\left(1+\lambda_k^2\right)}+(-1)^i\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4\left(1+\lambda_k^2\right)^2}-\frac{\beta}{1+\lambda_k^2}}\ (i=1,2),\\ \gamma_k=\mu_{2k}-\mu_{1k}=2\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4\left(1+\lambda_k^2\right)^2}-\frac{\beta}{1+\lambda_k^2}}\ .\end{gathered} \tag{25}$$

After substituting the expressions $u_{1k}(t) \quad (k=0,1,\ldots)$ and $u_{2k}(t) \quad (k=1,2,\ldots)$ into (18), for the component $u(x,t)$ of the solution $\{u(x,t),a(t)\}$ to the problem (1)–(3), (6), (7) we get

$$\begin{aligned} u(x,t) &= \varphi_{10} + t\psi_{10} + \int_0^t (t-\tau)F_{10}(\tau;u,a)d\tau + \\ &+ \sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[(\mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t})\varphi_{1k} + (e^{\mu_{2k}t} - e^{\mu_{1k}t})\psi_{1k} + \right.\right. \\ &\left.\left. + \int_0^t F_{1k}(\tau;u,a)(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)})d\tau\right]\right\}\cos\lambda_k x + \\ &+ \sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[(\mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t})\varphi_{2k} + (e^{\mu_{2k}t} - e^{\mu_{1k}t})\psi_{2k} + \right.\right. \\ &\left.\left. + \int_0^t F_{2k}(\tau;u,a)(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)})d\tau\right]\right\}\sin\lambda_k x. \end{aligned} \tag{26}$$

Now, from (7) and (18) we have

$$\begin{aligned} a(t) = [h(t)]^{-1}\{h''(t) - f(x_0,t) + \sum_{k=1}^{\infty}\left[\lambda_k^2 u_{1k}''(t) + \alpha\lambda_k^2 u_{1k}'(t) + \beta\lambda_k^2 u_{1k}(t)\right]\cos\lambda_k x_0 + \\ + \sum_{k=1}^{\infty}\left[\lambda_k^2 u_{2k}''(t) + \alpha\lambda_k^2 u_{2k}'(t) + \beta\lambda_k^2 u_{2k}(t)\right]\sin\lambda_k x_0\Bigg\}. \end{aligned} \tag{27}$$

Differentiating (24) twice, we get

$$\begin{aligned} u_{ik}'(t) = \frac{1}{\gamma_k}\Big[\mu_{1k}\mu_{2k}(e^{\mu_{1k}t} - e^{\mu_{2k}t})\varphi_{ik} + (\mu_{2k}e^{\mu_{2k}t} - \mu_{1k}e^{\mu_{1k}t})\psi_{ik} + \\ + \int_0^t F_{ik}(\tau;u,a)(\mu_{2k}e^{\mu_{2k}(t-\tau)} - \mu_{1k}e^{\mu_{1k}(t-\tau)})d\tau\Big] \quad (i=1,2), \end{aligned} \tag{28}$$

$$\begin{aligned} u_{ik}''(t) = \frac{1}{\gamma_k}\Big[\mu_{1k}\mu_{2k}(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t})\varphi_{ik} + (\mu_{2k}^2 e^{\mu_{2k}t} - \mu_{1k}^2 e^{\mu_{1k}t})\psi_{ik} + \\ + \int_0^t F_{ik}(\tau;u,a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)})d\tau\Big] + F_{ik}(t;u,a) \quad (i=1,2). \end{aligned} \tag{29}$$

By (20) and (29) we have

$$\begin{aligned} \lambda_k^2 u_{ik}''(t) + \alpha\lambda_k^2 u_{ik}'(t) + \beta\lambda_k^2 u_{ik}(t) = F_{ik}(t;u,a) - u_{ik}''(t) = \\ = -\frac{1}{\gamma_k}\Big[\mu_{1k}\mu_{2k}(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t})\varphi_{ik} + (\mu_{2k}^2 e^{\mu_{2k}t} - \mu_{1k}^2 e^{\mu_{1k}t})\psi_{ik} + \\ + \int_0^t F_{ik}(\tau;u,a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)})d\tau\Big] \quad (i=1,2). \end{aligned} \tag{30}$$

To obtain the equation for the second component $a(t)$ of the solution $\{u(x,t), a(t)\}$ to the problem (1)–(3), (6), (7), we substitute expression (30) into (27) and have

$$\begin{aligned} a(t) = [h(t)]^{-1}\{h''(t) - f(x_0,t) - \\ - \sum_{k=1}^{\infty}\frac{1}{\gamma_k}\Big[\mu_{1k}\mu_{2k}(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t})\varphi_{1k} + (\mu_{2k}^2 e^{\mu_{2k}t} - \mu_{1k}^2 e^{\mu_{1k}t})\psi_{1k} + \\ + \int_0^t F_{1k}(\tau;u,a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)})d\tau\Big]\cos\lambda_k x_0 - \\ - \sum_{k=1}^{\infty}\frac{1}{\gamma_k}\Big[\mu_{1k}\mu_{2k}(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t})\varphi_{2k} + (\mu_{2k}^2 e^{\mu_{2k}t} - \mu_{1k}^2 e^{\mu_{1k}t})\psi_{2k} + \\ + \int_0^t F_{2k}(\tau;u,a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)})d\tau\Big]\sin\lambda_k x_0\Bigg\}. \end{aligned} \tag{31}$$

Thus, the problem (1)–(3), (6), (7) is reduced to solving the system (26), (31) with respect to the unknown functions $u(x,t)$ and $a(t)$.

Similarly to [9], it is possible to prove the following lemma.

Lemma 2. *If* $\{u(x,t), a(t)\}$ *is any solution to the problem* (1)–(3), (6), (7), *then the functions*

$$u_{10}(t) = \int_0^1 u(x,t)dx,$$

$$u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx, \quad u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k = 1, 2, \ldots).$$

satisfy the system (23), (24) *in* $[0,T]$.

Remark. It follows from lemma 2 that to prove the uniqueness of the solution to the problem (1)–(3), (6), (7), it suffices to prove the uniqueness of the solution to the system (26), (31).

In order to investigate the problem (1) – (3), (6), (7), consider the following spaces

1. Denote by $B^3_{2,T}$ [8] the set of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x \quad (\lambda_k = 2\pi k),$$

defined on D_T such that the functions $u_{1k}(t)$ $(k = 0, 1, ...)$, $u_{2k}(t)$ $(k = 1, 2, ...)$ are continuous on $[0,T]$ and

$$J_T(u) \equiv \|u_{10}(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} < +\infty.$$

The norm on this set is given by

$$\|u(x,t)\|_{B^3_{2,T}} = J_T(u).$$

2. Denote by E^3_T the space $B^3_{2,T} \times C[0,T]$ of the vector-functions $z(x,t) = \{u(x,t), a(t)\}$ with the norm

$$\|z\|_{E^3_T} = \|u(x,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}.$$

It is known that $B^3_{2,T}$ and E^3_T are Banach spaces.

Now, in the space E^3_T consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x,$$

$$\Phi_2(u,a) = \tilde{a}(t),$$

$\tilde{u}_{10}(t)$, $\tilde{u}_{ik}(t)$, $k = 1, 2, \ldots)$ and $\tilde{a}(t)$ equal to the right hand sides of (23), (24), and (31), respectively.

It is easy to see that

$$\mu_{ik} < 0, \quad e^{\mu_{ik}t} \leqslant 1, \; e^{\mu_{ik}(t-\tau)} \leqslant 1 \, (i = 1, 2; k = 1, 2, ...; 0 \leqslant t \leqslant T, 0 \leqslant \tau \leqslant t),$$

$$|\mu_{ik}| \leqslant \lambda_k \left(\frac{\alpha\lambda_k}{2\left(1+\lambda_k^2\right)} + \sqrt{\frac{\alpha^2\lambda_k^2}{4\left(1+\lambda_k^2\right)^2} - \frac{\beta}{1+\lambda_k^2}} \right) \leqslant \frac{\alpha\lambda_k^2}{1+\lambda_k^2} \leqslant \alpha \left(i=1,2; k=1,2,...\right),$$

$$\mu_{1k}\mu_{2k} = \frac{\beta\lambda_k^2}{1+\lambda_k^2} \leqslant \beta,$$

$$\frac{1}{\gamma_k} = \frac{1}{2\sqrt{\frac{\lambda_k^2}{1+\lambda_k^2}\left(\frac{\alpha^2\lambda_k^2}{4\left(1+\lambda_k^2\right)} - \beta\right)}} \leqslant \frac{1}{2\sqrt{\frac{1}{2}\left(\frac{\alpha^2}{8} - \beta\right)}} \equiv \gamma_0 \left(k=1,2,...\right).$$

Taking into account these relations, by means of simple transformations we find

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} \leqslant |\varphi_{10}| + T\,|\psi_{10}| + T\sqrt{T}\left(\int_0^T |f_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}, \quad (32)$$

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} &\leqslant 4\alpha\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + 4\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \\ &+4\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + 4\gamma_0 T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}},\end{aligned} \quad (33)$$

$$\begin{aligned}\|\tilde{a}(t)\|_{C[0,T]} &\leqslant \left\|[\mathrm{h(t)}]^{-1}\right\|_{C[0,T]}\Big\{\|h''(t) - f(x_0,t)\|_{C[0,T]} + \\ &+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sum_{i=1}^{2}\Bigg[2\alpha\beta\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + 2\alpha^2\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \\ &+2\alpha^2\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \\ &+2\alpha^2\gamma_0\;T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Big\}.\end{aligned} \quad (34)$$

Suppose that the data of the problem (1)–(3), (6), (7) satisfy the following conditions

1. $\alpha > 0,\ \beta > 0,\ \dfrac{\alpha^2}{8} - \beta > 0.$
2. $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1), \varphi(0) = \varphi(1),\ \varphi'(0) = \varphi'(1),\ \varphi''(0) = \varphi''(1).$
3. $\psi(x) \in C^2[0,1],\ \psi'''(x) \in L_2(0,1), \psi(0) = \psi(1),\ \psi'(0) = \psi'(1),\ \psi''(0) = \psi''(1).$
4. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T),$
   $f(0,t) = f(1,t), f_x(0,t) = f_x(1,t), f_{xx}(0,t) = f_{xx}(1,t) \quad (0 \leqslant t \leqslant T).$
5. $h(t) \in C^2[0,T],\ h(t) \neq 0 \quad (0 \leqslant t \leqslant T).$

Then, from (32)–(34), we get

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leqslant A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \quad (35)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leqslant A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \quad (36)$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} +$$
$$+8\alpha\gamma_0\,\|\varphi'''(x)\|_{L_2(0,1)} + 8\gamma_0\,\|\psi'''(x)\|_{L_2(0,1)} + 8\gamma_0\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = T^2 + 8\gamma_0 T,$$

$$A_2(T) = \left\| [\mathrm{h(t)}]^{-1} \right\|_{C[0,T]} \Big\{ \|h''(t) - f(x_0,t))\|_{C[0,T]} +$$
$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \Big[4\alpha\beta\gamma_0 \left\| \varphi^{(3)}(x) \right\|_{L_2(0,1)} + 4\alpha^2\gamma_0 \left\| \psi^{(3)}(x) \right\|_{L_2(0,1)} +$$
$$+ \ 4\alpha^2\gamma_0\sqrt{T} \left\| f_{xxx}(x,t) \right\|_{L_2(D_T)} \Big] \Big\},$$

$$B_2(T) = 4\alpha^2\gamma_0 \left\| [\mathrm{h(t)}]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} T.$$

It follows from the inequalities (35), (36) that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leqslant A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^3_{2,T}}, \tag{37}$$

where $A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T)$.

Now we can prove the following theorem.

Theorem 1. *Let the conditions (1)–(5) be fulfilled and*

$$(A(T) + 2)^2 B(T) < 1. \tag{38}$$

Then the problem (1)–(3), (6), (7) *has a unique solution in the ball* $K = K_R(\|z\|_{E^3_T} \leqslant R = A(T) + 2)$ *of the space* E^3_T.

Proof. In the space E^3_T consider the equation

$$z = \Phi z, \tag{39}$$

where $z = \{u, a\}$ and the components $\Phi_i(u,a)$ $(i = 1,2)$ of the operator $\Phi(u,a)$ are given by the right hand sides of the equations (26), (31). Consider the operator $\Phi(u,a)$ in the ball $K = K_R$ from E^3_T. Similarly to (37), we see that for any $z, z_1, z_2 \in K_R$ the following estimates hold:

$$\|\Phi z\|_{E^3_T} \leqslant A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^3_{2,T}}, \tag{40}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leqslant B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}). \tag{41}$$

Then, it follows from (38) together with the estimates (40) and (41) that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u, a\}$, that is a unique solution to the equation (39), i.e. a unique solution to the system (26), (31).

The function $u(x,t)$, as an element of the space $B^3_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Now from (28) it is obvious that $u'_{ik}(t)(i = 1,2; k = 1,2,...)$ is continuous in $[0,T]$ and from the same relation we get

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u'_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leqslant 2\sqrt{2}\beta\gamma_0 \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + 2\sqrt{2}\alpha \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\psi_{ik}|)^2 \right)^{\frac{1}{2}} +$$
$$+ 2\alpha\sqrt{2T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_{ik}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + 2\sqrt{2}\alpha T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}},$$

or

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u'_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leqslant 2\sqrt{2}\beta\gamma_0\left\|\varphi'''(x)\right\|_{L_2(0,1)}+2\sqrt{2}\alpha\left\|\psi'''(x)\right\|_{L_2(0,1)}+$$

$$+2\alpha\sqrt{2T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+2\sqrt{2}\alpha T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}.$$

Hence, $u_t(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$ is continuous in D_T

Next, from (29) it follows that $u''_{ik}(t)(i=1,2;k=1,2,...)$ is continuous in $[0,T]$ and consequently we have:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u''_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leqslant 2\alpha\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u'_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+2\beta\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+2\left\|\left\|f(x,t)+a(t)u(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

From the last relation it is obvious that $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ is continuous in D_T.

It is easy to verify that the equation (1) and conditions (2), (3), (6), (7) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t)\}$ is a solution to the problem (1)–(3), (6), (7), and by Lemma 2 it is unique in the ball $K=K_R$. □

By Lemma 1 the unique solvability of the initial problem (1)–(5) follows from the theorem.

Theorem 2. *Let all the conditions of Theorem 1 be fulfilled and*

$$\int_0^1\varphi(x)dx=0,\quad \int_0^1\psi(x)dx=0,\quad \varphi(x_0)=h(0),\quad \psi(x_0)=h'(0).$$

Then the problem (1)–(5) *has a unique classical solution in the ball* $K=K_R(\|z\|_{E_T^3}\leqslant R=A(T)+2)$ *of the space* E_T^3.

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О разрешимости одной обратной краевой задачи для уравнения Бусинеска–Лява

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В работе исследована одна обратная краевая задача для уравнения Буссинеска–Лява. Сначала исходная задача сводится к эквивалентной в определенном смысле задаче. С помощью метода Фурье эквивалентная задача сводится к решению системы интегральных уравнений. Далее, с помощью метода сжатых отображений доказываются существование и единственность решения системы интегральных уравнений, которая также является единственным решением эквивалентной задачи. Пользуясь эквивалентностью, доказываются существование и единственность классического решения исходной задачи.

Ключевые слова: обратная краевая задача, уравнения Буссинеска–Лява, метод Фурье, классическое решение.