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Approximate Integration of Modified Riesz Potentials

Maria I. Medvedeva*
Nikolay N. Osipov†
Institute of Space and Information Technology,
Siberian Federal University,
Kirensky, 26, Krasnoyarsk, 660074
Russia

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The estimates for the errors of quadrature formulas are deduced in the case of integration of potentials such as the Feller potential, which is a modification of the Riesz potential.

Keywords: quadrature formula, error functional, sequence of functionals, approximate calculation, potential.

Introduction

The subject of this article was first studied in the papers [1, 2] where the authors considered functionals with a boundary layer on the space of functions that can be represented as the Riesz potentials. It was shown that such functionals have the best power rate of strong convergence among functionals with arbitrary nodes and coefficients as the number of nodes N increases indefinitely. In this paper we prove a similar result for functionals on functions that can be represented as the Feller potentials. In this case it is possible to show that the sequences of functionals with a boundary layer have asymptotically the best rate of convergence among the formulas with arbitrary nodes and coefficients.

The main results of this paper are Theorem 1 and 3. In Theorem 1 we derive upper bounds for the error functionals and lower bounds in Theorem 3. Lemmas 1, 2 and Theorem 2 are of auxiliary character.

Assume that a, b, p, q, α, and N are real numbers such that $a < b$, $1 < p < \infty$, $q = p/(p-1)$, $0 < \alpha < 1$, N is a positive integer. Suppose that $\alpha p > 1$ (in particular, this condition guarantees the inclusion of spaces considered here in the space $C[a, b]$ of continuous functions on $[a, b]$).

Let l^N be the error functional of the quadrature formula

$$(l^N, f) = \int_a^b f(x)\, dx - \sum_{k=1}^{N} c_k^N f(x_k^N), \tag{1}$$

where x_k^N ($x_k^N \in [a, b]$, $1 \leqslant k \leqslant N$) and c_k^N stand for the nodes and coefficients of the formula. In what follows, we shall assume that formula (1) is exact for constants.

Recall that the function $f(x)$ satisfies the Hölder condition on $[a, b]$ with the exponent λ, $0 < \lambda \leqslant 1$, if

$$|f(x_1) - f(x_2)| \leqslant K|x_1 - x_2|^\lambda$$

for all x_1, $x_2 \in [a, b]$, where K is a positive constant depending on the function $f(x)$ (for $\lambda = 1$ we have the Lipschitz condition). For a fixed λ, denote the set of such functions by $H^\lambda([a, b])$.

*mimedvedeva@rambler.ru
†nnosipov@rambler.r

Fix the number $\lambda > \alpha$. For the given α let

$$(B^{\alpha}\varphi)(x) = \int_a^b \frac{\operatorname{sgn}(x-t)\varphi(t)\,dt}{|x-t|^{1-\alpha}} \tag{2}$$

be the fractional integration operator. The equation

$$(B^{\alpha}\varphi)(x) = f(x)$$

is the special case ($u = v = 1$) of the general equation (see [3, p. 455, (30.79)])

$$u(I^{\alpha}_{a+}\varphi)(x) + v(I^{\alpha}_{b-}\varphi)(x) = f(x),$$

where u, v are constants and

$$(I^{\alpha}_{a+}\varphi)(x) = \int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt, \tag{3}$$

$$(I^{\alpha}_{b-}\varphi)(x) = \int_x^b (t-x)^{\alpha-1}\varphi(t)\,dt \tag{4}$$

for $a < x < b$.

Denote by $B^{\alpha}(L_p(a,b))$ the set of functions $f(x)$ on (a,b) of the form (2) with $\varphi(x) \in L_p(a,b)$. As usual, $L_p(a,b)$ is the linear space of measurable functions $\varphi : [a,b] \to \mathbb{R}$ with the norm

$$\|\varphi\|_{L_p(a,b)} = \left(\int_a^b |\varphi(x)|^p\,dx\right)^{1/p} < \infty.$$

Definition 1. *The numbers x and y from the segment $[a,b]$ are called dual if $x + y = a + b$.*

Definition 2. *The system of nodes $\theta = \{x_k^N : 1 \leqslant k \leqslant N\}$ of the quadrature formula*

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} c_k^N f(x_k^N)$$

is called symmetric if for each $x \in \theta$ its dual $y = a + b - x \in \theta$ too.

Definition 3. *A quadrature formula with a symmetric system of nodes is called symmetric if its coefficients at dual nodes are equal.*

Theorem 1. *There exist sequences of the error functionals for symmetric quadrature formulas $\{l^N\}$ (1) such that*

$$|(l^N, f)| \leqslant AN^{-\alpha}\|\varphi_f\|_{L_p(a,b)},$$

where $f = B^{\alpha}\varphi_f$ for $\varphi_f \in L_p(a,b)$, and A is a constant.

Proof. Let the sequence of functionals $\{l^N\}$ satisfy the conditions of the theorem for $f(x)$ of the form (3), i.e., $f(x) = (I^{\alpha}_{a+}\varphi_f)(x)$. The existence of such $\{l^N\}$ has been proved in [4] (in the case when $\alpha > 0$). Then such a sequence of functionals satisfies the theorem for $f(x)$ of the form (4), i.e., $f(x) = (I^{\alpha}_{b-}\varphi_f)(x)$. To see that, replace in the integral (4) the variable t by $a + bt$

$$\int_x^b (t-x)^{\alpha-1}\varphi(t)dt = \int_a^{a+b-x} (a+b-x-t)^{\alpha-1}\varphi(a+b-t)dt$$

and take into account the symmetry of the quadrature formulas corresponding to l^N, i.e.,

$$\int_a^b dx(I^{\alpha}_{b-}\varphi)(x) \approx \sum_{k=1}^{N} c_k^N \int_a^{y_k^N} (y_k^N - t)^{\alpha-1}\varphi(a+b-t)dt,$$

where $y_k^N = a + b - x_k^N$ is the dual point for x_k^N. □

Remark. The statement of the theorem is satisfied by a number of well-known sequences of quadrature formulas such as the (complicated) Gregory formula (see [4]).

Consider the integral equation

$$\int_a^b \frac{\operatorname{sgn}(x-t)}{|x-t|^{1-\alpha}}\varphi_f(t)\,dt = f(x), \qquad 0<\alpha<1, \tag{5}$$

which is an important special case of the generalized Abel equation on a segment. Its left hand side is called the Feller potential. The equation (5) is solvable for any right hand side from $H^\lambda([a,b])$ and has a nontrivial solution of the form

$$\begin{aligned}\varphi_f(x) = {} & \frac{c}{(x-a)^{(1+\alpha)/2}(b-x)^{(1+\alpha)/2}} + \frac{1}{2\pi}\operatorname{ctg}\frac{\alpha\pi}{2}\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha} - \frac{\cos^2\frac{\alpha\pi}{2}}{2\pi^2}\times \\ & \times\frac{d}{dx}\left(\int_a^b [(t-a)(b-t)]^{1-\alpha/2} f(t)\,dt \int_a^b \frac{\operatorname{sgn}(x-y)\,dy}{(t-y)|x-y|^\alpha[(y-a)(b-y)]^{1-\alpha/2}} - \right. \\ & \left. - \int_a^b [(t-a)(b-t)]^{(1-\alpha)/2} f(t)\,dt \int_a^b \frac{\operatorname{sgn}(x-t)\,dy}{(t-y)|x-y|^\alpha[(y-a)(b-y)]^{(1-\alpha)/2}}\right)\end{aligned} \tag{6}$$

with an arbitrary constant c (see [3, p. 457–459]). Further on we shall take $c=0$.

Lemma 1. *If $f(x)\in H^\lambda([a,b])$ satisfies $f(a)=f(b)=0$, then the corresponding function $\varphi_f(x)$ from (5) can be represented in the form*

$$\varphi_f(x) = \frac{c}{(x-a)^{(1+\alpha)/2}(b-x)^{(1+\alpha)/2}} + \frac{\sin\alpha\pi}{2\pi}\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha}. \tag{7}$$

Proof. Let $f(x)\in H^\lambda([a,b])$ for some $\lambda>\alpha$. Using the equalities (see [5], c. 530–531)

$$\int_a^b \frac{\operatorname{sgn}(x-y)\,dy}{(t-y)|x-y|^\alpha[(y-a)(b-y)]^{1-\alpha/2}} = \frac{\pi\operatorname{ctg}\frac{\alpha\pi}{2}}{|t-x|^\alpha[(t-a)(b-t)]^{1-\alpha/2}},$$

$$\int_a^b \frac{\operatorname{sgn}(x-y)\,dy}{(t-y)|x-y|^\alpha[(y-a)(b-y)]^{(1-\alpha)/2}} = \frac{(-1)\pi\operatorname{ctg}\frac{1+\alpha}{2}\pi}{|x-t|^\alpha[(t-a)(b-t)]^{(1-\alpha)/2}},$$

$0<\alpha<1,\ a<x<b,\ a<t<b$, we calculate the interior integrals in the second and third summands of (6).

Then the expression (6) turns into

$$\begin{aligned}\varphi_f(x) = {} & \frac{c}{(x-a)^{(1+\alpha)/2}(b-x)^{(1+\alpha)/2}} + \frac{1}{2\pi}\operatorname{ctg}\frac{\alpha\pi}{2}\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha} - \\ & - \frac{\cos^2\frac{\alpha\pi}{2}}{2\pi^2}\frac{d}{dx}\left(\pi\operatorname{ctg}\frac{\alpha\pi}{2}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha} + \pi\operatorname{ctg}\frac{1+\alpha}{2}\pi\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha}\right) = \\ = {} & \frac{c}{(x-a)^{(1+\alpha)/2}(b-x)^{(1+\alpha)/2}} + \frac{1}{2\pi}\operatorname{ctg}\frac{\alpha\pi}{2}\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha} - \\ & - \frac{1}{2\pi}\cos^2\frac{\alpha\pi}{2}\left(\operatorname{ctg}\frac{\alpha\pi}{2} - \operatorname{tg}\frac{\alpha\pi}{2}\right)\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha} = \\ = {} & \frac{c}{(x-a)^{(1+\alpha)/2}(b-x)^{(1+\alpha)/2}} + \frac{\sin\alpha\pi}{2\pi}\frac{d}{dx}\int_a^b \frac{f(t)\,dt}{|x-t|^\alpha}.\end{aligned}$$

□

Put $h = h(N) = (b-a)/(2N)$, $\Omega(N) = \bigcup\limits_{i\in\sigma(N)} (a+hi, a+hi+h)$, where $\sigma(N)$ is the family of integers $i \in [0, 2N-1]$ such that the intervals $(a+hi, a+hi+h)$ do not contain the nodes $x_1^N, \ldots, x_N^N$ of formula (1).

Let the function $\psi_h(x)$ belong to $C([0,2N])$, vanish outside the segment $[0,2N]$, and satisfy

$$\psi_h(x) = \begin{cases} \psi(x-i), & i \in \sigma(N),\ x \in (i, i+1); \\ 0, & i \notin \sigma(N), \end{cases}$$

where $\psi(x) \in C([0,1])$, $\psi(0) = \psi(1) = 0$,

$$\int_0^1 \psi(x)\,dx = 0,$$

and is equal to zero outside $[0,1]$.

In what follows, the symbol k with different indices denotes positive constants independent of h, a, b.

Let

$$\mathcal{B}_{d_1,d_2}(z) = \int_{d_1}^{d_2} \frac{\psi_h(s)\,ds}{|z-s|^\alpha},$$

where d_1, d_2 and z some numbers such that $0 \leqslant d_1 < d_2$ and $0 \leqslant z \leqslant 2N$.

Lemma 2. *There exists a constant k_1 independent of z such that*

$$|\mathcal{B}_{0,2N}(z)| < k_1.$$

Proof. In Lemma 18 of [4] it was proved that the integral

$$\mathcal{B}_{0,z}(z) = \int_0^z \frac{\psi_h(s)ds}{(z-s)^\alpha}$$

is bounded

$$|\mathcal{B}_{0,z}(z)| < k_2. \tag{8}$$

Note that the functions

$$(I_{a+}^\alpha \varphi)(x) = \int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt$$

can be transformed to

$$(I_{b-}^\alpha \varphi)(x) = \int_x^b (t-x)^{\alpha-1}\varphi(t)\,dt$$

by a simple change of variables (see [3], p. 42)

$$QI_{a+}^\alpha = I_{b-}^\alpha Q, \qquad QI_{b-}^\alpha = I_{a+}^\alpha Q, \qquad (Q\varphi)(x) = \varphi(a+b-x).$$

Thus, for the integral $\mathcal{B}_{z,2N}(z)$ there is a similar estimate, namely,

$$|\mathcal{B}_{z,2N}(z)| = \left| \int_z^{2N} \frac{\psi_h(s)ds}{(s-z)^\alpha} \right| < k_3.$$

The statement of lemma follows from this inequality, the relations (8) and the following equality

$$\mathcal{B}_{0,2N}(z) = \int_0^z \frac{\psi_h(s)ds}{(z-s)^\alpha} + \int_z^{2N} \frac{\psi_h(s)ds}{(s-z)^\alpha}.$$

□

Consider a continuously differentiable function $g(x) \in H^{\lambda}([0,1])$ such that $g(0) = g(1) = g'(0) = g'(1) = 0$ and

$$\int_0^1 g(x)dx > 0.$$

Extend $g(x)$ to the whole real axis by setting it equal to zero outside the interval $[0,1]$ and denote

$$g_h(x) = \begin{cases} g\left(\frac{x-a}{h} - i\right), & i \in \sigma(N),\ x \in (a+hi, a+hi+h), \\ 0, & i \notin \sigma(N). \end{cases} \tag{9}$$

By construction, the continuously differentiable function $g_h(x)$ belongs to the space $H^{\lambda}([a,b])$.

Theorem 2. *Let the function $g_h(x)$ be defined by formula (9). Then there exist functions $\varphi_g^h(x)$ such that*

$$g_h(x) = \int_a^b |x-t|^{\alpha-1}\varphi_g^h(t)\,dt, \qquad 0 < \alpha < 1, \tag{10}$$

and

$$\|\varphi_g^h\|_{L_p(a,b)} \leqslant k_4 h^{-\alpha}, \tag{11}$$

where k_4 is a positive constant.

Proof. Since $g_h(x)$ belongs to the space $H^{\lambda}([a,b])$ and $g_h(a) = g_h(b) = 0$, the inversion formula (7) with $f(x) = g_h(x)$ holds. It is known (see [3, p. 43]) that if a function f is continuously differentiable on the segment $[a,b]$, then

$$\begin{aligned} \frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha}} &= \frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)\,dt}{(x-t)^{\alpha}}, \\ \frac{d}{dx}\int_x^b \frac{f(t)\,dt}{(t-x)^{\alpha}} &= \frac{f(b)}{(b-x)^{\alpha}} - \int_x^b \frac{f'(t)\,dt}{(t-x)^{\alpha}}. \end{aligned}$$

Then from the last equations and formulas (7), (10) with $c = 0$ and relations

$$(g_h)'(x) = \begin{cases} h^{-1} g'\left(\frac{x-a}{h} - i\right), & i \in \sigma(N),\ x \in (a+hi, a+hi+h), \\ 0, & i \notin \sigma(N) \end{cases}$$

we infer that

$$\begin{aligned} |\varphi_g^h(x)| &= k_5\left|\int_a^b \frac{(g_h)'(t)\,dt}{|x-t|^{\alpha}}\right| = k_5 h^{-1}\left|\sum_{i\in\sigma(N)}\int_{a+ih}^{a+ih+h} \frac{g'\left(\frac{t-a}{h} - i\right)dt}{|x-t|^{\alpha}}\right| = \\ &= k_5\left|\sum_{i\in\sigma(N)}\int_i^{i+1} \frac{g'(\tau-i)\,d\tau}{|x-h\tau-a|^{\alpha}}\right| = k_5 h^{-\alpha}\left|\sum_{i\in\sigma(N)}\int_i^{i+1} \frac{g'(\tau-i)\,d\tau}{\left|\frac{x-a}{h}-\tau\right|^{\alpha}}\right|. \end{aligned}$$

Applying Lemma 2, with $\psi_h(x) = g'(x)$ and $z = (x-a)/h$, to the right hand side of the expression above, we obtain that

$$|\varphi_g^h(x)| \leqslant k_6 h^{-\alpha}.$$

Theorem 2 is completely proved. □

Theorem 3. *For every sequence of functionals of the form (1) there exist a number $k_7 > 0$ and functions $\varphi_f(x) \in L_p(a,b)$ such that*

$$|(l^N, f)| > k_7 N^{-\alpha}\|\varphi_f\|_{L_p(a,b)}(b-a)^{1/q+\alpha}.$$

Proof. Let $g_h(x)$ be the function from formula (9). Since $g_h(x_k^N) = 0$ $(1 \leqslant k \leqslant N)$, we infer

$$(l^N, g_h) = \int_a^b g_h(x)\,dx = \int_{\Omega(N)} g_h(x)\,dx =$$
$$= \sum_{i\in\mu(N)} \int_{a+ih}^{a+ih+h} g\left(\frac{x-a}{h} - i\right) dx = \operatorname{mes}\Omega(N) \int_0^1 g(x)dx \geqslant$$
$$\geqslant \frac{(b-a)}{2} \int_0^1 g(x)dx > 0. \quad (12)$$

Using (11) and (12), we find

$$|(l^N, g_h)| > k_8 (b-a)^{1/q} h^\alpha \|\varphi_g\|_{L_p(a,b)} > k_9 N^{-\alpha} (b-a)^{1/q+\alpha} \|\varphi_g\|_{L_p(a,b)},$$

where k_8 and $k_9 > 0$ are some constants. Theorem 3 is proved. □

Combining Theorems 1 and 3, we can formulate the following result: there exist sequences of points $\{x_k^N\}_{k=1}^N \subset [a,b]$, numbers $\{c_k^N\}_{k=1}^N$, constants A, $B > 0$, and functions of the form (2) such that for the error functionals (1) the following inequality holds

$$BN^{-\alpha} \|\varphi_f\|_{L_p(a,b)} (b-a)^{1/q+\alpha} < |(l^N, f)| < AN^{-\alpha} \|\varphi_f\|_{L_p(a,b)} (b-a)^{1/q+\alpha}, 0 < \alpha < 1.$$

Thus, these sequences give the error functionals for quadrature formulas with the best rate of convergence to zero (on functions of the form (2)) as the number of nodes N increases indefinitely.

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Приближенное интегрирование модифицированных потенциалов Рисса

Мария И. Медведева
Николай Н. Осипов

Выводятся оценки погрешностей квадратурных формул при интегрировании потенциалов типа потенциала Феллера, представляющих собой модификацию потенциалов Рисса.

Ключевые слова: квадратурная формула, функционал ошибки, последовательность функционалов, приближенное вычисление, потенциал.