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On the Solvability of one Class of Boundary-value Problems for Non-linear Integro-differential Equation in Kinetic Theory of Plazma

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The work is devoted to the investigation of one class of non-linear integro-differential equations with the Hammerstein non-compact operator on the half-line. The mentioned class of equations has direct application in the kinetic theory of plazma. Combining the special factorization methods with the theory of construction of invariant cone intervals for non-linear operators permits to prove the existence of a solution of the initial equation in the Sobolev space $W_1^1(\mathbb{R}^+)$.

Keywords: factorization, kernel, monotonicity, iteration, Caratheodory's condition, Sobolev space.

Introduction

We consider the following boundary-value problem for the nonlinear integro-differential equation of the Hammerstein type non-compact operator

$$\begin{cases} -\dfrac{df}{dx} = \mu \displaystyle\int_0^\infty \{K(x-t) - \varepsilon K(x+t)\} h(t, f(t))dt, \quad x \geqslant 0, & (1) \\ f(+\infty) \equiv \displaystyle\lim_{x\to\infty} f(x) = 0, & (2) \end{cases}$$

with respect to a measurable real-valued function $f(x)$. Here $\mu > 0$ and $\varepsilon \in [0, 1)$ are positive numerical parameters of equation (1), and the kernel $K(x)$ has the following form:

$$K(x) = \int_a^b e^{-|x|s} G(s)ds, \quad x \in \mathbb{R} \equiv (-\infty, +\infty), \tag{3}$$

where $G(s)$ is the positive continuous and monotonically decreasing function on $[a, b)$ ($a > 0$, $b > a\sqrt{3}$, $b \leqslant +\infty$); besides,

$$2\int_a^b \frac{G(s)}{s} ds = 1, \tag{4}$$

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$h(t,z)$ is defined on set $\mathbb{R}^+ \times \mathbb{R}$ ($\mathbb{R}^+ \equiv [0,+\infty)$), takes real values and satisfies the condition of criticity:

$$h(t,0) \equiv 0, \quad \forall t \in \mathbb{R}^+. \tag{5}$$

The problem (1)–(2) has direct application in the kinetic theory of plazma (see [1–3]). In particular, equation (1) is used to describe the problem of stationar distribution of electrons in semi-infinite plazma, where the role of function $f(x)$ plays the first coordinate of the electric field $\vec{E}(x) = (f(x), 0, 0)$, and ε is the coefficient of the accommodation.

Equation (1) is derived from the Boltzmann model equation taking into the consideration the energy interaction in the integral of collision (see [3]).

In case where $\varepsilon = 0$, $G(s) = \dfrac{1}{s^2}$, $a = 1$, $b = +\infty$, the problem (1)–(2) has been studied in [2] by one of the authors of the present paper. In this article the existence of a positive solution in the Sobolev space $W_1^1(\mathbb{R}^+)$ is proved under some additional assumptions on the function $h(t,z)$.

In the present paper, under suitable assumptions on h, we construct a positive solution of problem (1)–(2) in the space $W_1^1(\mathbb{R}^+)$; in addition the structure of the solution is described. A list of examples of the function $h(t,z)$ is given at the end of the paper.

1. Reduction of the problem (1)–(2) to integral equation

Integrating the both sides of equation (1) from a positive number τ to $+\infty$ and using (2) we get

$$f(\tau) = \int_0^\infty \{T(\tau - t) - T_0(\tau + t)\} h(t, f(t)) dt, \quad \tau > 0, \tag{6}$$

where

$$T(r) = \mu \int_r^{+\infty} K(x) dx, \quad -\infty < r < +\infty, \tag{7}$$

$$T_0(r) = \varepsilon\mu \int_r^{+\infty} K(x) dx, \quad 0 < r < +\infty. \tag{8}$$

Note that

$$T(r) \notin L_1(-\infty, +\infty), \tag{9}$$

since conditions (3) and (4) imply

$$T(-\infty) = \mu > 0.$$

On the other hand

$$T_0 \in L_1(0, +\infty). \tag{10}$$

Really, this fact follows from Fubini's theorem because

$$\int_0^\infty x K(x) dx = \int_a^b \frac{G(s)}{s^2} ds \leqslant \frac{1}{2a} < +\infty. \tag{11}$$

We introduce the following functions

$$T_\alpha(x) = e^{\alpha x} T(x) > 0, \quad x \in \mathbb{R}, \tag{12}$$

$$T_0^\alpha(x) = e^{\alpha x} T_0(x) > 0, \quad x \in \mathbb{R}^+, \tag{13}$$

where $\alpha > 0$. It follows from formulas (7) and (8) that

$$T_\alpha(x) = \begin{cases} \mu \displaystyle\int_a^b e^{-(s-\alpha)x} \frac{G(s)}{s} ds, & \text{for } x \geqslant 0, \\ \mu e^{\alpha x} - \mu \displaystyle\int_a^b e^{(s+\alpha)x} \frac{G(s)}{s} ds, & \text{for } x \leqslant 0, \end{cases} \in L_1(-\infty, +\infty), \tag{14}$$

$$T_0^\alpha(x) = \varepsilon\mu \int_a^b e^{-(s-\alpha)x} \frac{G(s)}{s} ds \in L_1(0,+\infty), \tag{15}$$

if

$$\alpha \in (0, a). \tag{16}$$

Everywhere below, unless otherwise stated, we assume that (16) is fulfilled.

To find the parameter α we impose the following conservativity condition on the kernel $T_\alpha(x)$:

$$\int_{-\infty}^{+\infty} T_\alpha(x)dx = 1. \tag{17}$$

Using (17), we obtain to the following characteristic equation with respect to α:

$$\frac{\mu}{\alpha} \int_a^b \frac{2s}{s^2-\alpha^2} G(s)ds = 1. \tag{18}$$

Consider the function

$$\mu = \tilde{\mu}(\alpha) = \frac{\alpha}{\int_a^b \frac{2s}{s^2-\alpha^2} G(s)ds}, \quad \alpha \in (0, a). \tag{19}$$

Note that

$$1) \quad \tilde{\mu}(\alpha) > 0, \quad \alpha \in (0, a), \tag{20}$$

$$2) \quad \tilde{\mu} \in C(0, a), \tag{21}$$

$$3) \quad \tilde{\mu}(0^+) \equiv \lim_{\alpha\to 0^+} \tilde{\mu}(\alpha) = 0. \tag{22}$$

The function $\tilde{\mu}(\alpha)$ is strongly increasing on $(0, \alpha_0]$ and strongly decreasing on $[\alpha_0, a)$, where the maximum point α_0 is determined from the following equation

$$\int_a^b \frac{s(s^2-3\alpha^2)}{(s^2-\alpha^2)^2} G(s)ds = 0. \tag{23}$$

Let us check, that equation (23) has a unique solution on the interval $(0, a)$. In fact, due to the assumptions imposed on G, the function

$$\chi(\alpha) = \int_a^b \frac{s(s^2-3\alpha^2)}{(s^2-\alpha^2)^2} G(s)ds, \quad \alpha \in (0, a), \tag{24}$$

possesses the following properties

$$a)\ \chi(0^+) = \lim_{\alpha\to 0^+} \chi(\alpha) = \frac{1}{2}, \qquad b)\ \chi'(\alpha) < 0, \quad \alpha \in (0, a), \qquad c)\ \chi(a^-) = \lim_{\alpha\to a^-} \chi(\alpha) = -\infty.$$

Taking into the consideration formula (4), the properties *a*) and *b*) immediately follow from (24).

In order to prove property *c*) we note that

$$\chi(\alpha) = \int_a^{a\sqrt{3}} \frac{s(s^2-3\alpha^2)}{(s^2-\alpha^2)^2} G(s)ds + \int_{a\sqrt{3}}^b \frac{s(s^2-3\alpha^2)}{(s^2-\alpha^2)^2} G(s)ds \equiv I_1(\alpha) + I_2(\alpha),$$

$$0 \leqslant I_2(\alpha) \leqslant \int_{a\sqrt{3}}^b \frac{s(s^2-3\alpha^2)}{(s^2-\alpha^2)^2} G(s)ds \leqslant \frac{3}{2} \int_a^b \frac{G(s)}{s} ds = \frac{3}{4},$$

(because $3\alpha^2 \leqslant 3a^2 \leqslant s^2$ on the integration set of the integral $I_2(\alpha)$).

Now let us check that $I_1(a^-) = -\infty$ or $\lim_{\alpha\to a^-} (-I_1(\alpha)) = +\infty$.

We have

$$-I_1(\alpha)=\int_a^{a\sqrt{3}}\frac{s(3\alpha^2-s^2)}{(s^2-\alpha^2)^2}G(s)ds\geqslant G(a\sqrt{3})\int_a^{a\sqrt{3}}\frac{s(3\alpha^2-s^2)}{(s^2-\alpha^2)^2}ds=$$

$$=G(a\sqrt{3})\left(\frac{1}{2}\ln(a^2-\alpha^2)+\frac{\alpha^2}{a^2-\alpha^2}-\frac{\alpha^2}{3a^2-\alpha^2}-\frac{1}{2}\ln(3a^2-\alpha^2)\right)\geqslant$$

$$\geqslant G(a\sqrt{3})\left(\ln\left(\frac{\alpha^2}{a^2-\alpha^2}+1\right)+\ln\sqrt{a^2-\alpha^2}-\frac{\alpha^2}{3a^2-\alpha^2}-\ln\sqrt{3a^2-\alpha^2}\right)=$$

$$=G(a\sqrt{3})\left(\ln\frac{a^2}{\sqrt{a^2-\alpha^2}}-\frac{\alpha^2}{3a^2-\alpha^2}-\ln\sqrt{3a^2-\alpha^2}\right)\underset{\alpha\to a^-}{\longrightarrow}+\infty.$$

Therefore $\chi(a^-)=-\infty$. Thus it follows from properties $a)-c)$ that equation (23) on $(0,a)$ has a unique solution $\alpha=\alpha_0$.

Then on intervals $(0,\alpha_0]$ and $[\alpha_0,a)$ there exist inverse functions of $\tilde{\mu}(\alpha)$, which we denote by $\alpha_1(\mu)$ and $\alpha_2(\mu)$ respectively.

We calculate the value of the first moment of the kernel T_α:

$$\nu=\nu(T_\alpha)=\frac{\mu}{\alpha^2}\int_a^b\frac{s(3\alpha^2-s^2)}{(s^2-\alpha^2)^2}G(s)ds. \tag{25}$$

Using the above mentioned facts and taking into the consideration formula (25) we get the following Lemma:

Lemma 1. *Let the conditions* (3), (4), (16) *and* (17) *be fulfilled. Then*
I) *if* $\nu(T_\alpha)<0$, *then* $\alpha=\alpha_1(\mu)$, $\mu<\tilde{\mu}(\alpha_0)$,
II) *if* $\nu(T_\alpha)>0$, *then* $\alpha=\alpha_2(\mu)$, $\mu<\tilde{\mu}(\alpha_0)$,
III) *if* $\nu(T_\alpha)=0$, *then* $\alpha=\alpha_0$, $\mu=\tilde{\mu}(\alpha_0)$.

Let the conditions (17), (16) be fulfilled and $\nu(T_\alpha)<0$. Then multiplying the both sides of (6) by function $e^{\alpha_1(\mu)x}$, we get the following equation

$$\psi(\tau)=e^{\alpha_1(\mu)\tau}\int_0^\infty\{T(\tau-t)-T_0(\tau+t)\}h(t,e^{-\alpha_1(\mu)t}\psi(t))dt \tag{26}$$

with respect to the function

$$\psi(x)=e^{\alpha_1(\mu)x}f(x). \tag{27}$$

In the following sections we will deal with the solutions of the equation (26).

2. On one convolution type auxiliary equation

We consider the following convolution type linear integral equation on the half-line with respect to measurable function $S(x)$:

$$S(x)=\int_0^\infty\{T_{\alpha_1}(x-t)-e^{-2\alpha_1 t}T_0^{\alpha_1}(x+t)\}S(t)dt,\quad x\geqslant 0 \tag{28}$$

where $\alpha_1=\alpha_1(\mu)$. We rewrite equation (28) in the operator form in the Banach space E (E being $L_p(0,+\infty)$, $1\leqslant p<+\infty$ or $M(0,+\infty)\equiv L_\infty(0,+\infty)$ or $C_0(0,+\infty)$):

$$(I-\hat{T}_{\alpha_1}+\hat{T}_0^{\alpha_1})S=0, \tag{29}$$

where I is the identity operator, $\hat{T}_{\alpha_1}$ is the Wiener-Hopf operator with the conservative kernel $T_{\alpha_1}(x)$:

$$(\hat{T}_{\alpha_1} f)(x) = \int_0^\infty T_\alpha(x-t)f(t)dt, \quad f \in E \tag{30}$$

and operator $\hat{T}_0^{\alpha_1}$ is given in according by formula:

$$(\hat{T}_0^{\alpha_1} f)(x) = \int_0^\infty e^{-2\alpha_1 t} T_0^{\alpha_1}(x+t)f(t)dt, \quad f \in E. \tag{31}$$

It is well known that the operators (30) acting in the Banach space E are non-compact and they permit the following Volteryan factorization under the assumption (17) (see [4]):

$$I - \hat{T}_{\alpha_1} = (I - \hat{V}_-^{\alpha_1})(I - \hat{V}_+^{\alpha_1}), \tag{32}$$

where

$$(\hat{V}_-^{\alpha_1} f)(x) = \int_x^\infty v_-^{\alpha_1}(t-x)f(t)dt, \qquad (\hat{V}_+^{\alpha_1} f)(x) = \int_0^x v_+^{\alpha_1}(x-t)f(t)dt, \tag{33}$$

$f \in E$, $v_\pm^{\alpha_1}(x) \geqslant 0, \quad x \in \mathbb{R}^+, \ v_\pm^{\alpha_1} \in L_1(0,+\infty)$. Besides if $\nu(\hat{T}_{\alpha_1}) < 0$ then

$$\gamma_+^{\alpha_1} = \int_0^\infty v_+^{\alpha_1}(x)dx < 1, \quad \gamma_-^{\alpha_1} = \int_0^\infty v_-^{\alpha_1}(x)dx = 1. \tag{34}$$

The factorization (32) is understood as the coincidence of the operators, acting in E.

Unlike the Wiener-Hopf operators, the operators of the type (31) are completely continuous for the indicated above choice of the space E. We denote by Ω_{α_1} the class of these operators, i.e. $\hat{B}_{\alpha_1} \in \Omega_{\alpha_1}$ if and only if

$$(\hat{B}_{\alpha_1} f)(x) = \int_0^\infty e^{-2\alpha_1 t} B_{\alpha_1}(x+t)f(t)dt, \quad f \in E, \tag{35}$$

$B_{\alpha_1} \in L_1(0,+\infty)$.

We consider the following factorization problem: given operators $\hat{T}_{\alpha_1}$ and $\hat{T}_0^{\alpha_1}$ (see formula (30), (31)), find such an operator $\hat{B}_{\alpha_1} \in \Omega_{\alpha_1}$, that the following factorization holds:

$$I - \hat{T}_{\alpha_1} + \hat{T}_0^{\alpha_1} = (I - \hat{V}_-^{\alpha_1})(I + \hat{B}_{\alpha_1})(I - \hat{V}_+^{\alpha_1}), \tag{36}$$

where operators $\hat{V}_\pm^{\alpha_1}$ are defined by (33).

Here the factorization is also understood as coincideness of the operators acting in E.

The factorization is constructed step by step. First we consider the following factorization

$$I - \hat{T}_{\alpha_1} + \hat{T}_0^{\alpha_1} = (I - \hat{V}_-^{\alpha_1})(I - \hat{V}_+^{\alpha_1} + \hat{U}_{\alpha_1}), \tag{37}$$

where operator $\hat{U}_{\alpha_1}$ is sought in Ω_{α_1}. Due to (32), decomposition (37) is equivalent to the equality

$$\hat{U}_{\alpha_1} = \hat{T}_0^{\alpha_1} + \hat{V}_-^{\alpha_1}\hat{U}_{\alpha_1}. \tag{38}$$

Using the operator equality (38) we come to the coincideness of the corresponding kernels. Then, after the calculations, we obtain

$$U_{\alpha_1}(x) = T_0^{\alpha_1}(x) + \int_0^\infty v_-^{\alpha_1}(t)U_{\alpha_1}(x+t)dt, \quad x \geq 0, \tag{39}$$

where $U_{\alpha_1}(x+t)e^{-2\alpha_1 t}$–is the kernel of integral operator $\hat{U}_{\alpha_1}$.

Since $\gamma_-^{\alpha_1}=1$, $T_0^{\alpha_1}\in L_1(0,+\infty)$ and $\int_0^\infty xT_0^{\alpha_1}(x)dx<+\infty$, then it follows from the results of [4] that equation (39) has a positive solution in $L_1(0,+\infty)$.

Therefore factorization (37) exists.

At the second step we are looking for the factorization of operator $I-\hat{V}_+^{\alpha_1}+\hat{U}_{\alpha_1}$ in the form

$$I-\hat{V}_+^{\alpha_1}+\hat{U}_{\alpha_1}=(I+\hat{B}_{\alpha_1})(I-\hat{V}_+^{\alpha_1}). \tag{40}$$

We note that factorization (40) is equivalent to the equation

$$B_{\alpha_1}(x)=U_{\alpha_1}(x)+\int_0^\infty B_{\alpha_1}(x+t)\tilde{v}_+^{\alpha_1}(t)dt,\quad x\geqslant 0, \tag{41}$$

where $\tilde{v}_+^{\alpha_1}(t)=e^{-2\alpha_1 t}v_+^{\alpha_1}(t)$, and $B_{\alpha_1}(x+t)e^{-2\alpha_1 t}$ is the kernel of the integral operator $\hat{B}_{\alpha_1}\in\Omega_{\alpha_1}$.

In fact, using (40) it is easy to check that for arbitrary function $f\in E$ we have

$$\int_0^\infty B_{\alpha_1}(x+t)e^{-2\alpha_1 t}f(t)dt=$$

$$=\int_0^\infty U_{\alpha_1}(x+t)e^{-2\alpha_1 t}f(t)dt+\int_0^\infty B_{\alpha_1}(x+t)e^{-2\alpha_1 t}\int_0^t v_+^{\alpha_1}(t-y)f(y)dydt.$$

Changing the order of the integration with the use of Fubin's theorem we obtain

$$\int_0^\infty B_{\alpha_1}(x+t)e^{-2\alpha_1 t}f(t)dt=$$

$$=\int_0^\infty U_{\alpha_1}(x+t)e^{-2\alpha_1 t}f(t)dt+\int_0^\infty f(y)\int_y^\infty B_{\alpha_1}(x+t)e^{-2\alpha_1 t}v_+^{\alpha_1}(t-y)f(y)dtdy.$$

From (41) we get

$$B_{\alpha_1}(x+t)e^{-2\alpha_1 t}=U_{\alpha_1}(x+t)e^{-2\alpha_1 t}+\int_t^{+\infty}B_{\alpha_1}(x+z)e^{-2\alpha_1 z}v_+^{\alpha_1}(z-t)dz.$$

Replacing the variable of the integration, $z-t=\tau$, we obtain an equation, equivalent to (41):

$$B_{\alpha_1}(x+t)e^{-2\alpha_1 t}=U_{\alpha_1}(x+t)e^{-2\alpha_1 t}+\int_0^\infty B_{\alpha_1}(x+t+\tau)e^{-2\alpha_1(t+\tau)}v_+^{\alpha_1}(\tau)d\tau.$$

We rewrite equation (41) in the following from

$$B_{\alpha_1}(x)=U_{\alpha_1}(x)+\int_x^\infty \tilde{v}_-^{\alpha_1}(t-x)B_{\alpha_1}(t)dt,\quad x\geqslant 0. \tag{42}$$

As $U_{\alpha_1}\in L_1(0,+\infty)$, $\tilde{v}_+^{\alpha_1}\geqslant 0$ and $\int_0^\infty \tilde{v}_+^{\alpha_1}(x)dx\leqslant\gamma_+^{\alpha_1}<1$, then equation (42) has a positive solution in $L_1(0,+\infty)$.

Therefore there exists factorization (40). Combining (38) and (40) we get (36).

Taking into account (36), equation (29) yields

$$(I-\hat{V}_-^{\alpha_1})(I+\hat{B}_{\alpha_1})(I-\hat{V}_+^{\alpha_1})S=0. \tag{43}$$

The solution of equation (43) is reduced to the successive solution of the following coupled operators equations

$$(I-\hat{V}_-^{\alpha_1})F=0, \tag{44}$$

$$(I + \hat{B}_{\alpha_1})\varphi = F, \tag{45}$$

$$(I - \hat{V}_+^{\alpha_1})S = \varphi. \tag{46}$$

We consider the following possible cases for the completely continuous operator $\hat{B}_{\alpha_1}$:

1) " -1" is the characteristic value for the operator $\hat{B}_{\alpha_1}$

2) " -1" is the not characteristic value for the operator $\hat{B}_{\alpha_1}$.

Case 1): We consider the homogeneous equation (44). As the solution of (44) we take the trivial solution $F(x) \equiv 0$. Since "-1" is the characteristic value of the operator $\hat{B}_{\alpha_1}$, then equation $\hat{B}_{\alpha_1}\varphi = -\varphi$ has a non-trivial bounded solution (because $\hat{B}_{\alpha_1}$ is the completely continuous operator in space $M(0, +\infty)$). Let us turn to the solution of the equation (46) with respect to the function S:

$$S(x) = \varphi(x) + \int_0^x v_+^{\alpha_1}(x-t)S(t)dt, \quad x \geqslant 0, \tag{47}$$

where $\varphi \in M(0, +\infty)$.

Since $\gamma_+^{\alpha_1} < 1$, then equation (47) has a unique solution $S \in M(0, +\infty)$ in space of bounded functions.

Case 2): As a solution of equation (44) we take $F(x) \equiv 1$ (it is possible because $\gamma_-^{\alpha_1} = 1$). Since "-1" is not a characteristic value of the operator $\hat{B}_{\alpha_1}$, then due to the complete continuity of the operator $\hat{B}_{\alpha_1}$ in the space $M(0, +\infty)$ we conclude that there exists the bounded inverse operator $(I + \hat{B}_{\alpha_1})^{-1}$ (see [5]). Since $F \in M(0, +\infty)$, then $\varphi \in M(0, +\infty)$. We again come to equation (47), which has bounded solution.

Thus, it has been proved that equation (28) has a non-trivial bounded solution $S(x)$. However, the solution may alternate in sign. Below we prove that equation (28) possesses also a non-trivial non-negative monotonically increasing bounded solution $S^*(x)$. With this purpose, first let us check that

$$T_{\alpha_1}(x-t) \geqslant e^{-2\alpha_1 t} T_0^{\alpha_1}(x+t), \quad \forall (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{48}$$

Really taking into account (7), (8), (12) and (13) we have

$$T_{\alpha_1}(x-t) = \mu e^{\alpha_1(x-t)} \int_{x-t}^{\infty} K(y)dy \geqslant \mu\varepsilon e^{-2\alpha_1 t} e^{\alpha_1(x+t)} \int_{x+t}^{\infty} K(y)dy = e^{-2\alpha_1 t} T_0^{\alpha_1}(x+t).$$

Now we introduce the following iterations (successive approximations) for equation (28):

$$\begin{gathered} S_{n+1}(x) = \int_0^\infty \{T_{\alpha_1}(x-t) - e^{-2\alpha_1 t} T_0^{\alpha_1}(x+t)\} S_n(t)dt, \quad x \geqslant 0, \\ S_0(x) = \sup_{\tau \geqslant 0} |S(\tau)|, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{49}$$

where $S(x)$ is the bounded solution of equation (28), constructed by means of the factorization (36).

By the induction, taking into consideration (48), it is easy to check that

$$i) \quad S_n(x) \downarrow \text{ with respect to } n, \qquad ii) \quad S_n(x) \geqslant |S(x)|, \quad n = 0, 1, 2, \ldots. \tag{50}$$

Therefore the sequence of functions $\{S_n(x)\}_{n=0}^{\infty}$ has the pointwise limit $\lim_{n\to\infty} S_n(x) = S^*(x) \leqslant \sup_{x \geqslant 0} |S(x)|$. Moreover, in accordance with B.Levi's theorem (see [6]), this limit satisfies the following inequalities

$$|S(x)| \leqslant S^*(x) \leqslant \sup_{x \geqslant 0} |S(x)|, \quad x \in \mathbb{R}^+. \tag{51}$$

We show that $S^*(x)\uparrow$ while x. With this aim, we rewrite the iterations (49) in the following form

$$S_{n+1}(x)=\int_{-\infty}^{x}T_{\alpha_1}(\tau)S_n(x-\tau)d\tau-\int_0^\infty e^{-2\alpha_1 t}T_0^{\alpha_1}(x+t)S_n(t)dt,\qquad S_0(x)=\sup_{x\geqslant 0}|S(x)|,\quad n=0,1,2,\ldots. \tag{52}$$

Taking into consideration (15), (52), and using the induction, it is easy to prove that

$$S_n(x)\uparrow \text{ with respect to } x,\quad n=0,1,2,\ldots. \tag{53}$$

Therefore $S^*(x)\uparrow$ with respect to x. Thus the following theorem holds.

Theorem 1. *Let the conditions* (17), (16) *be fulfilled and* $\nu(T_\alpha)<0$. *Then equation* (28) *possesses a non-negative non-trivial monotonically increasing bounded solution* $S^*(x)$.

In the following paragraph, using theorem 1, we will prove the existence theorem for the problem (1)–(2) in space the $W_1^1(0,+\infty)$.

3. Solvability of problem (1)–(2)

The following theorem is true.

Theorem 2. *Let the conditions* (3)–(5) *be fulfilled. Assume that there exists* $\eta>0$, *such that*

j_1) $h(t,z)\uparrow$ *with respect to* z *for each fixed* $t\in\mathbb{R}^+$ *on the interval* $[0,\eta e^{-\alpha_1(\mu)t}]$, *where* $\alpha_1(\mu)$ *is uniquely defined by relation* (17) *while* $\nu(T_\alpha)<0$,

j_2) $h(t,z)$ *satisfies Caratheodory's condition on the interval* $\mathbb{R}^+\times[0,\eta]$ *with respect to the arguments* z, *i.e. for each fixed* $z\in[0,\eta]$ *the function* $h(t,z)$ *is measurable with respect to* $t\in\mathbb{R}^+$ *and it is continuous with respect to* z *on the interval* $[0,\eta]$ *for almost all* $t\in\mathbb{R}^+$,

j_3) $h(t,z)\geqslant z,\ t\geqslant 0,\ z\in[0,\eta e^{-\alpha_1(\mu)t}]$,

j_4) $h(t,\eta e^{-\alpha_1(\mu)t})=\eta e^{-\alpha_1(\mu)t}$.

Then the problem (1)–(2) has a non-trivial non-negative bounded solution of the following structure

$$f(x)=e^{-\alpha_1(\mu)x}\psi(x), \tag{54}$$

in the space $W_1^1(\mathbb{R}^+)$ *where* $\psi\in W_\infty^1(\mathbb{R}^+)\equiv\{\varphi:\varphi^{(j)}\in L_\infty(0,+\infty),\ j=0,1\}$,

$$0\leqslant\psi(x)\leqslant\eta,\quad \psi(x)\not\equiv 0,\ x\in\mathbb{R}^+. \tag{55}$$

Proof. Consider the following successive approximations for the equation (26):

$$\psi_{n+1}(\tau)=e^{\alpha_1(\mu)\tau}\int_0^\infty\{T(\tau-t)-T_0(\tau+t)\}h(t,e^{-\alpha_1(\mu)t}\psi_n(t))dt, \tag{56}$$

$\psi_0(\tau)\equiv\eta,\ n=0,1,2,\ldots,\quad \tau\geqslant 0.$

By the induction, we can easily prove that

$$\psi_n(\tau)\downarrow \text{ with respect to } n. \tag{57}$$

Indeed,

$$\psi_1(\tau)=e^{\alpha_1(\mu)\tau}\int_0^\infty\{T(\tau-t)-T_0(\tau+t)\}h(t,e^{-\alpha_1(\mu)t}\eta)dt\leqslant\eta\int_0^\infty T_{\alpha_1}(\tau-t)dt\leqslant\eta,$$

since $\int_{-\infty}^{+\infty} T_{\alpha_1}(z)dz = 1$,

$$\psi_1(\tau) \geqslant e^{\alpha_1(\mu)\tau} \int_0^\infty \{T(\tau - t) - T_0(\tau + t)\}h(t, 0)dt = 0.$$

We assume that $0 \le \psi_n(\tau) \leqslant \psi_{n-1}(\tau)$ for some $n \in \mathbb{N}$. Then, due to condition j_1), equation (56) implies

$$\psi_{n+1}(\tau) \ge 0 \text{ и } \psi_{n+1}(\tau) \leqslant e^{\alpha_1(\mu)\tau} \int_0^\infty \{T(\tau - t) - T_0(\tau + t)\}h(t, e^{-\alpha_1(\mu)t}\psi_{n-1}(t))dt = \psi_n(\tau).$$

Now we prove that

$$\psi_n(\tau) \geqslant \frac{S^*(\tau)}{\sup\limits_{t\geqslant 0} S^*(t)}\eta,\ \tau \geqslant 0, \quad n = 0, 1, 2, \dots. \tag{58}$$

If $n = 0$ then inequality (58) is obviously fulfilled. Let (58) holds true for some $n \in \mathbb{N}$. Then due to conditions j_3), equation (56) yields

$$\psi_{n+1}(\tau) \geqslant e^{\alpha_1(\mu)\tau} \int_0^\infty \{T(\tau - t) - T_0(\tau + t)\}h\left(t, \eta e^{-\alpha_1(\mu)t}\frac{S^*(t)}{\sup\limits_{t\geqslant 0} S^*(t)}\right)dt \geqslant$$
$$\geqslant \frac{\eta}{\sup\limits_{t\geqslant 0} S^*(t)} \int_0^\infty \{T_{\alpha_1(\mu)}(\tau - t) - e^{-2\alpha_1(\mu)t}T_0^{\alpha_1(\mu)}(\tau + t)\}S^*(t)dt = \frac{S^*(\tau)}{\sup\limits_{t\geqslant 0} S^*(t)}\eta,$$

where $\alpha_1 \equiv \alpha_1(\mu)$.

Thus, the sequence of functions $\{\psi_n(\tau)\}_{n=0}^\infty$ has a pointwise limit $\lim\limits_{n\to\infty} \psi_n(\tau) = \psi(\tau)$ while $n \to \infty$; besides $\psi(\tau)$ satisfies the following chain of the inequalities

$$\frac{\eta|S(\tau)|}{\sup\limits_{t\geqslant 0} S^*(t)} \leqslant \frac{\eta S^*(\tau)}{\sup\limits_{t\geqslant 0} S^*(t)} \leqslant \psi(\tau) \leqslant \eta, \quad \tau \geqslant 0. \tag{59}$$

Taking into the consideration B.Levi's theorem and condition j_2) we conclude that the limit function $\psi(\tau)$ satisfies equation (26). From formula (27) we obtain the following inequalities:

$$\frac{\eta e^{-\alpha_1(\mu)x} S^*(x)}{\sup\limits_{t\geqslant 0} S^*(t)} \leqslant f(x) \leqslant \eta e^{-\alpha_1(\mu)x}, \quad x \geqslant 0. \tag{60}$$

Now let us check that $\psi \in W_\infty^1(0, +\infty)$. In fact, $H(x) \equiv \frac{dT}{dx}$, $\overset{\circ}{H}(x) \equiv \frac{dT_0}{dx}$ are continuous and integrable functions on the sets $(-\infty; +\infty)$ and $(0, +\infty)$ respectively (because of functions T and T_0 are given by (7) and (8), and K is the exponential function of the type (3)). Besides, the integrals

$$e^{\alpha_1(\mu)x} \int_0^\infty H(x - t)h(t, e^{-\alpha_1(\mu)t}\psi(t))dt,$$
$$e^{\alpha_1(\mu)x} \int_0^\infty \overset{\circ}{H}(x + t)h(t, e^{-\alpha_1(\mu)t}\psi(t))dt,$$
$$e^{\alpha_1(\mu)x} \int_0^\infty T(x - t)h(t, e^{-\alpha_1(\mu)t}\psi(t))dt,$$
$$e^{\alpha_1(\mu)x} \int_0^\infty T_0(x + t)h(t, e^{-\alpha_1(\mu)t}\psi(t))dt$$

uniformly (bounded) converge, and $\psi \in M(0,+\infty)$. Hence, due to the theorems on the differentiation of the parameter dependent integral (see [7]), we conclude that $\psi' \in M(0,+\infty)$. Therefore the theorem has been proved. □

Remark 1. *It should be noted that in particular case where $h(t,z) = z$ (it satisfies conditions $j_1) - j_4)$) the solution of problem (1)-(2) reduces to the solution of equation* (28). *But since the solution of equation* (28) *is bounded function in the case where $\nu(T_\alpha) < 0$ only (in other cases, using results of [4], it can be proved that non-trivial solutions are unbounded). On the other hand, we are looking for bounded solutions to equation* (26). *Therefore we have to assume that $\nu(T_\alpha) < 0$.*

Remark 2. *It is also interesting to note that if $h(t,z)$ satisfies conditions $j_1) - j_4)$, and*
1) $h(t,0) \not\equiv 0, \quad t \in \mathbb{R}^+$,
2) *there exists number $L \in (0,1)$, such that*

$$|h(t,z_1) - h(t,z_2)| \leqslant L|z_1 - z_2|, \quad t \in \mathbb{R}^+, \quad z_1, z_2 \in [0, \eta e^{-\alpha_1(\mu)t}],$$

then it is easy to prove the uniqueness of the solution to problem (1)-(2) *in $W_1^1(\mathbb{R}^+)$.*

An example of the function $h(t,z)$ satisfying all the assumptions above is the following:

$$h(t,z) = z - \frac{\eta e^{\alpha_1(\mu)t}}{2} + \frac{\eta^2 e^{-2\alpha_1(\mu)t}}{z + \eta e^{-\alpha_1(\mu)t}}.$$

Here the number L can be chosen as follows: $L = \dfrac{3}{4}$.

At the end of paper we give a number of examples for the function $h(t,z)$:

$$a) \quad h(t,z) = \sqrt{\eta e^{-\alpha_1(\mu)t} z},$$

$$b) \quad h(t,z) = z + \frac{\eta e^{-\alpha_1(\mu)t}}{\pi} \sin^2 \frac{\pi z}{\eta e^{-\alpha_1(\mu)t}},$$

$$c) \quad h(t,z) = \sqrt{\eta e^{-\alpha_1(\mu)t} z e^{\frac{z e^{\alpha_1(\mu)t}}{\eta} - 1}}.$$

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О разрешимости одного класса граничных задач нелинейных интегро-дифференциальных уравнений в кинетической теории плазмы

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Работа посвящена исследованию одного класса нелинейных интегро-дифференциальных уравнений с некомпактным оператором Гаммерштейна в полуплоскости. Рассматриваемый класс уравнений имеет прямое приложение в кинетической теории плазмы. Комбинация методов специальной факторизации с теорией построения инвариантных конусных отрезков для нелинейных операторов позволила доказать существование решения рассматриваемых уравнений в пространствах Соболева.

Ключевые слова: факторизация, ядро, монотонность, итерация, условие Каратеодори, пространство Соболева.