On the Solvability of one Class of Boundary-value Problems for Non-linear Integro-differential Equation in Kinetic Theory of Plazma

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The work is devoted to the investigation of one class of non-linear integro-differential equations with the Hammerstein non-compact operator on the half-line. The mentioned class of equations has direct application in the kinetic theory of plasma. Combining the special factorization methods with the theory of construction of invariant cone intervals for non-linear operators permits to prove the existence of a solution of the initial equation in the Sobolev space $W^1_1(\mathbb{R}^+)$. 

**Keywords:** factorization, kernel, monotonicity, iteration, Caratheodory’s condition, Sobolev space.

**Introduction**

We consider the following boundary-value problem for the nonlinear integro-differential equation of the Hammerstein type non-compact operator

\[
\begin{align*}
-\frac{df}{dx} &= \mu \int_0^\infty \{K(x-t) - \varepsilon K(x+t)\}h(t, f(t))dt, \quad x \geq 0, \\
f(+\infty) &= \lim_{x \to +\infty} f(x) = 0, \\
K(x) &= \int_a^b e^{-|x|s}G(s)ds, \quad x \in \mathbb{R} \equiv (-\infty, +\infty), \\
2 \int_a^b \frac{G(s)}{s}ds &= 1,
\end{align*}
\]

with respect to a measurable real-valued function $f(x)$. Here $\mu > 0$ and $\varepsilon \in [0, 1)$ are positive numerical parameters of equation (1), and the kernel $K(x)$ has the following form:

where $G(s)$ is the positive continuous and monotonically decreasing function on $[a, b)$ ($a > 0$, $b > a\sqrt{3}$, $b \leq +\infty$); besides,

\[2 \int_a^b \frac{G(s)}{s}ds = 1,\]
$h(t, z)$ is defined on set $\mathbb{R}^+ \times \mathbb{R}$ ($\mathbb{R}^+ \equiv [0, +\infty)$), takes real values and satisfies the condition of criticity:

$$h(t, 0) \equiv 0, \quad \forall t \in \mathbb{R}^+. \quad (5)$$

The problem (1)–(2) has direct application in the kinetic theory of plasma (see [1–3]). In particular, equation (1) is used to describe the problem of stationary distribution of electrons in semi-infinite plasma, where the role of function $f(x)$ plays the first coordinate of the electric field $\vec{E}(x) = (f(x), 0, 0)$, and $\varepsilon$ is the coefficient of the accommodation.

Equation (1) is derived from the Boltzmann model equation taking into the consideration the energy interaction in the integral of collision (see [3]).

In case where $\varepsilon = 0$, $G(s) = \frac{1}{s^2}$, $a = 1$, $b = +\infty$, the problem (1)–(2) has been studied in [2] by one of the authors of the present paper. In this article the existence of a positive solution in the Sobolev space $W^{1}_{1}(\mathbb{R}^+)$ is proved under some additional assumptions on the function $h(t, z)$.

In the present paper, under suitable assumptions on $h$, we construct a positive solution of problem (1)–(2) in the space $W^{1}_{1}(\mathbb{R}^+)$; in addition the structure of the solution is described. A list of examples of the function $h(t, z)$ is given at the end of the paper.

1. Reduction of the problem (1)–(2) to integral equation

Integrating the both sides of equation (1) from a positive number $\tau$ to $+\infty$ and using (2) we get

$$f(\tau) = \int_{0}^{\infty} \{T(\tau - t) - T_0(\tau + t)\} h(t, f(t)) dt, \quad \tau > 0, \quad (6)$$

where

$$T(r) = \mu \int_{r}^{+\infty} K(x) dx, \quad -\infty < r < +\infty, \quad (7)$$
$$T_0(r) = \varepsilon \mu \int_{r}^{+\infty} K(x) dx, \quad 0 < r < +\infty. \quad (8)$$

Note that

$$T(r) \notin L_1(-\infty, +\infty), \quad (9)$$

since conditions (3) and (4) imply

$$T(-\infty) = \mu > 0. \quad (10)$$

On the other hand

$$T_0 \in L_1(0, +\infty). \quad (10)$$

Really, this fact follows from Fubini’s theorem because

$$\int_{0}^{\infty} x K(x) dx = \int_{a}^{b} \frac{G(s)}{s^2} ds \leq \frac{1}{2a} < +\infty. \quad (11)$$

We introduce the following functions

$$T_{\alpha}(x) = e^{\alpha x} T(x) > 0, \quad x \in \mathbb{R}, \quad (12)$$
$$T_{\alpha}^0(x) = e^{\alpha x} T_0(x) > 0, \quad x \in \mathbb{R}^+, \quad (13)$$

where $\alpha > 0$. It follows from formulas (7) and (8) that

$$T_{\alpha}(x) = \begin{cases} 
\mu \int_{a}^{b} e^{-s-\alpha x} \frac{G(s)}{s} ds, & \text{for } x \geq 0, \\
\mu \varepsilon \alpha^{\alpha x} - \mu \int_{a}^{b} e^{s+\alpha x} \frac{G(s)}{s} ds, & \text{for } x \leq 0,
\end{cases} \quad (14)$$
Using (17), we obtain to the following characteristic equation with respect to

$$T_0^\alpha(x) = \varepsilon \mu \int_a^b e^{-s^2-\alpha^2 s} G(s) \frac{ds}{s} \in L_1(0, +\infty), \quad (15)$$

if

$$\alpha \in (0, a). \quad (16)$$

Everywhere below, unless otherwise stated, we assume that (16) is fulfilled.

To find the parameter $\alpha$ we impose the following conservativity condition on the kernel $T_\alpha(x)$:

$$\int_{-\infty}^{+\infty} T_\alpha(x) dx = 1. \quad (17)$$

Using (17), we obtain to the following characteristic equation with respect to $\alpha$:

$$\frac{\mu}{\alpha} \int_a^b \frac{2s}{s^2-\alpha^2} G(s) ds = 1. \quad (18)$$

Consider the function

$$\mu = \tilde{\mu} = \frac{\alpha}{\int_a^b \frac{2s}{s^2-\alpha^2} G(s) ds}, \quad \alpha \in (0, a). \quad (19)$$

Note that

1) $\tilde{\mu}(\alpha) > 0, \quad \alpha \in (0, a), \quad (20)$

2) $\tilde{\mu} \in C(0, a), \quad (21)$

3) $\tilde{\mu}(0^+) \equiv \lim_{\alpha \to 0^+} \tilde{\mu}(\alpha) = 0. \quad (22)$

The function $\tilde{\mu}(\alpha)$ is strongly increasing on $(0, \alpha_0]$ and strongly decreasing on $[\alpha_0, a)$, where the maximum point $\alpha_0$ is determined from the following equation

$$\int_a^b \frac{s^2 - 3\alpha^2}{(s^2 - \alpha^2)^2} G(s) ds = 0. \quad (23)$$

Let us check, that equation (23) has a unique solution on the interval $(0, a)$. In fact, due to the assumptions imposed on $G$, the function

$$\chi(\alpha) = \int_a^b \frac{s(s^2 - 3\alpha^2)}{(s^2 - \alpha^2)^2} G(s) ds, \quad \alpha \in (0, a), \quad (24)$$

possesses the following properties

a) $\chi(0^+) = \lim_{\alpha \to 0^+} \chi(\alpha) = \frac{1}{2}, \quad b) \chi'(\alpha) < 0, \quad \alpha \in (0, a), \quad c) \chi(a^-) = \lim_{\alpha \to a^-} \chi(\alpha) = -\infty.$

Taking into the consideration formula (4), the properties a) and b) immediately follow from (24).

In order to prove property c) we note that

$$\chi(\alpha) = \int_a^{a\sqrt{3}} \frac{s(s^2 - 3\alpha^2)}{(s^2 - \alpha^2)^2} G(s) ds + \int_{a\sqrt{3}}^{b} \frac{s(s^2 - 3\alpha^2)}{(s^2 - \alpha^2)^2} G(s) ds \equiv I_1(\alpha) + I_2(\alpha),$$

where

$$0 \leq I_2(\alpha) \leq \int_a^{b} \frac{s(s^2 - 3\alpha^2)}{(s^2 - \alpha^2)^2} G(s) ds \leq \frac{3}{2} \int_a^b \frac{G(s)}{s} ds = \frac{3}{4}.$$ 

(because $3\alpha^2 \leq s^2 \leq a^2$ on the integration set of the integral $I_2(\alpha)$).

Now let us check that $I_1(a^-) = -\infty$ or $\lim_{\alpha \to a^-} (I_1(\alpha)) = +\infty$. 

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We have

\[-I_1(\alpha) = \int_a^{\alpha \sqrt{3}} \frac{s(3\alpha^2 - s^2)}{(s^2 - \alpha^2)^2} G(s)ds \geq G(\alpha \sqrt{3}) \int_a^{\alpha \sqrt{3}} \frac{s(3\alpha^2 - s^2)}{(s^2 - \alpha^2)^2} ds =
\]

\[= G(\alpha \sqrt{3}) \left( \frac{1}{2} \ln(a^2 - \alpha^2) + \frac{\alpha^2}{a^2 - \alpha^2} - \frac{\alpha^2}{3a^2 - \alpha^2} - \frac{1}{2} \ln(3a^2 - \alpha^2) \right) \geq
\]

\[\geq G(\alpha \sqrt{3}) \left( \ln \left( \frac{\alpha^2}{a^2 - \alpha^2} + 1 \right) + \ln \sqrt{a^2 - \alpha^2} - \frac{\alpha^2}{3a^2 - \alpha^2} - \ln \sqrt{3a^2 - \alpha^2} \right) =
\]

\[= G(\alpha \sqrt{3}) \left( \ln \frac{\alpha^2}{\sqrt{a^2 - \alpha^2}} - \frac{\alpha^2}{3a^2 - \alpha^2} - \ln \sqrt{3a^2 - \alpha^2} \right) \rightarrow +\infty.
\]

Therefore \(\chi(a^-) = -\infty\). Thus it follows from properties a) – c) that equation (23) on \((0, a)\) has a unique solution \(\alpha = \alpha_0\).

Then on intervals \([0, \alpha_0]\) and \([\alpha_0, a]\) there exist inverse functions of \(\bar{\mu}(\alpha)\), which we denote by \(\alpha_1(\mu)\) and \(\alpha_2(\mu)\) respectively.

We calculate the value of the first moment of the kernel \(T_0\):

\[\nu = \nu(T_0) = \frac{\mu}{\alpha^2} \int_a^b \frac{s(3\alpha^2 - s^2)}{(s^2 - \alpha^2)^2} G(s)ds.
\] (25)

Using the above mentioned facts and taking into the consideration formula (25) we get the following Lemma:

**Lemma 1.** Let the conditions (3), (4), (16) and (17) be fulfilled. Then

I) if \(\nu(T_0) < 0\), then \(\alpha = \alpha_1(\mu), \quad \mu < \bar{\mu}(\alpha_0)\),

II) if \(\nu(T_0) > 0\), then \(\alpha = \alpha_2(\mu), \quad \mu < \bar{\mu}(\alpha_0)\),

III) if \(\nu(T_0) = 0\), then \(\alpha = \alpha_0, \quad \mu = \bar{\mu}(\alpha_0)\).

Let the conditions (17), (16) be fulfilled and \(\nu(T_0) < 0\). Then multiplying the both sides of (6) by function \(e^{\alpha_1(\mu)x}\), we get the following equation

\[\psi(t) = e^{\alpha_1(\mu)x} \int_0^\infty \left\{ T(t - t) - T_0(t + t) \right\} h(t, e^{-\alpha_1(\mu)t}\psi(t))dt
\] (26)

with respect to the function

\[\psi(x) = e^{\alpha_1(\mu)x} f(x).
\] (27)

In the following sections we will deal with the solutions of the equation (26).

2. **On one convolution type auxiliary equation**

We consider the following convolution type linear integral equation on the half-line with respect to measurable function \(S(x)\):

\[S(x) = \int_0^\infty \{ T_0(x - t) - e^{-2\alpha_1 t} T_0^{\alpha_1}(x + t) \} S(t)dt, \quad x \geq 0
\] (28)

where \(\alpha_1 = \alpha_1(\mu)\). We rewrite equation (28) in the operator form in the Banach space \(E\) (\(E\) being \(L_p(0, +\infty)\), \(1 \leq p < +\infty\) or \(M(0, +\infty) \equiv L_\infty(0, +\infty)\) or \(C_0(0, +\infty)\)):

\[(I - \hat{T}_\alpha + \hat{T}_0^{\alpha_1})S = 0,
\] (29)
where \( I \) is the identity operator, \( \hat{T}_{\alpha_1} \) is the Wiener-Hopf operator with the conservative kernel \( T_{\alpha_1}(x) \):

\[
(\hat{T}_{\alpha_1}f)(x) = \int_0^\infty T_{\alpha_1}(x-t)f(t)dt, \quad f \in E
\]

and operator \( \hat{T}_{\alpha_1}^0 \) is given in accordance by formula:

\[
(\hat{T}_{\alpha_1}^0 f)(x) = \int_0^\infty e^{-2\alpha_1 t}T_{\alpha_1}^0(x+t)f(t)dt, \quad f \in E.
\]

It is well known that the operators (30) acting in the Banach space \( E \) are non-compact and they permit the following Volteryan factorization under the assumption (17) (see [4]):

\[
I - \hat{T}_{\alpha_1} = (I - \hat{V}_{\alpha_1}^+)(I - \hat{V}_{\alpha_1}^-),
\]

where

\[
(\hat{V}_{\alpha_1}^+ f)(x) = \int_x^\infty v_{\alpha_1}^+(t-x)f(t)dt, \quad (\hat{V}_{\alpha_1}^- f)(x) = \int_0^x v_{\alpha_1}^+(x-t)f(t)dt,
\]

\( f \in E, \ v_{\alpha_1}^\pm(x) \geq 0, \ x \in \mathbb{R}^+, \ v_{\alpha_1}^\pm \in L_1(0, +\infty). \) Besides if \( \nu(\hat{T}_{\alpha_1}) < 0 \) then

\[
\gamma_+^{\alpha_1} = \int_0^\infty v_{\alpha_1}^+(x)dx < 1, \quad \gamma_-^{\alpha_1} = \int_0^\infty v_{\alpha_1}^-(x)dx = 1.
\]

The factorization (32) is understood as the coincidence of the operators, acting in \( E \).

Unlike the Wiener-Hopf operators, the operators of the type (31) are completely continuous for the indicated above choice of the space \( E \). We denote by \( \Omega_{\alpha_1} \) the class of these operators, i.e. \( \hat{B}_{\alpha_1} \in \Omega_{\alpha_1} \) if and only if

\[
(\hat{B}_{\alpha_1}f)(x) = \int_0^\infty e^{-2\alpha_1 t}B_{\alpha_1}(x+t)f(t)dt, \quad f \in E,
\]

\( B_{\alpha_1} \in L_1(0, +\infty). \)

We consider the following factorization problem: given operators \( \hat{T}_{\alpha_1} \) and \( \hat{T}_{\alpha_1}^0 \) (see formula (30), (31)), find such an operator \( \hat{B}_{\alpha_1} \in \Omega_{\alpha_1} \), that the following factorization holds:

\[
I - \hat{T}_{\alpha_1} + \hat{T}_{\alpha_1}^0 = (I - \hat{V}_{\alpha_1}^+)(I + \hat{B}_{\alpha_1})(I - \hat{V}_{\alpha_1}^-),
\]

where operators \( \hat{V}_{\alpha_1}^\pm \) are defined by (33).

Here the factorization is also understood as coincidence of the operators acting in \( E \).

The factorization is constructed step by step. First we consider the following factorization

\[
I - \hat{T}_{\alpha_1} + \hat{T}_{\alpha_1}^0 = (I - \hat{V}_{\alpha_1}^+)(I - \hat{V}_{\alpha_1}^- + \hat{U}_{\alpha_1}),
\]

where operator \( \hat{U}_{\alpha_1} \) is sought in \( \Omega_{\alpha_1} \). Due to (32), decomposition (37) is equivalent to the equality

\[
\hat{U}_{\alpha_1} = \hat{V}_{\alpha_1}^- + \hat{V}_{\alpha_1}^+ \hat{U}_{\alpha_1}.
\]

Using the operator equality (38) we come to the coincidence of the corresponding kernels. Then, after the calculations, we obtain

\[
U_{\alpha_1}(x) = T_{\alpha_1}^0(x) + \int_0^\infty v_{\alpha_1}^+(t)U_{\alpha_1}(x+t)dt, \quad x \geq 0,
\]

where \( U_{\alpha_1}(x+t)e^{-2\alpha_1 t} \) is the kernel of integral operator \( \hat{U}_{\alpha_1} \).
Since $\gamma^{\alpha_1} = 1$, $T_0^{\alpha_1} \in L_1(0, +\infty)$ and $\int_0^\infty xT_0^{\alpha_1}(x) \, dx < +\infty$, then it follows from the results of [4] that equation (39) has a positive solution in $L_1(0, +\infty)$.

Therefore factorization (37) exists.

At the second step we are looking for the factorization of operator $I - \hat{\mathcal{V}}_+^{\alpha_1} + \hat{\mathcal{U}}_{\alpha_1}$ in the form

$$I - \hat{\mathcal{V}}_+^{\alpha_1} + \hat{\mathcal{U}}_{\alpha_1} = (I + \hat{\mathcal{B}}_{\alpha_1})(I - \hat{\mathcal{V}}_+^{\alpha_1}).$$  

(40)

We note that factorization (40) is equivalent to the equation

$$B_{\alpha_1}(x) = U_{\alpha_1}(x) + \int_0^\infty B_{\alpha_1}(x + t)\tilde{\mathcal{V}}_+^{\alpha_1}(t) \, dt, \quad x \geq 0,$$

(41)

where $\tilde{\mathcal{V}}_+^{\alpha_1}(t) = e^{-2\alpha_1 t}v_+^{\alpha_1}(t)$, and $B_{\alpha_1}(x + t)e^{-2\alpha_1 t}$ is the kernel of the integral operator $B_{\alpha_1} \in \Omega_{\alpha_1}$.

In fact, using (40) it is easy to check that for arbitrary function $f \in E$ we have

$$\int_0^\infty B_{\alpha_1}(x + t)e^{-2\alpha_1 t} f(t) \, dt = \int_0^\infty U_{\alpha_1}(x + t)e^{-2\alpha_1 t} f(t) \, dt + \int_0^\infty B_{\alpha_1}(x + t)e^{-2\alpha_1 t} \int_0^t \tilde{\mathcal{V}}_+^{\alpha_1}(t - y) f(y)\, dy \, dt.$$

Changing the order of the integration with the use of Fubin’s theorem we obtain

$$\int_0^\infty B_{\alpha_1}(x + t)e^{-2\alpha_1 t} f(t) \, dt = \int_0^\infty U_{\alpha_1}(x + t)e^{-2\alpha_1 t} f(t) \, dt + \int_0^\infty f(y) \int_0^\infty B_{\alpha_1}(x + t)e^{-2\alpha_1 t} \tilde{\mathcal{V}}_+^{\alpha_1}(t - y) f(y) \, dt \, dy.$$

From (41) we get

$$B_{\alpha_1}(x + t)e^{-2\alpha_1 t} = U_{\alpha_1}(x + t)e^{-2\alpha_1 t} + \int_t^\infty B_{\alpha_1}(x + z)e^{-2\alpha_1 z}v_+^{\alpha_1}(z - t) \, dz.$$

Replacing the variable of the integration, $z - t = \tau$, we obtain an equation, equivalent to (41):

$$B_{\alpha_1}(x + t)e^{-2\alpha_1 t} = U_{\alpha_1}(x + t)e^{-2\alpha_1 t} + \int_0^\infty B_{\alpha_1}(x + t + \tau)e^{-2\alpha_1 (t + \tau)}v_+^{\alpha_1}(\tau) \, d\tau.$$

We rewrite equation (41) in the following from

$$B_{\alpha_1}(x) = U_{\alpha_1}(x) + \int_x^\infty \tilde{\mathcal{V}}_+^{\alpha_1}(t - x) B_{\alpha_1}(t) \, dt, \quad x \geq 0.$$  

(42)

As $U_{\alpha_1} \in L_1(0, +\infty)$, $\tilde{\mathcal{V}}_+^{\alpha_1} \geq 0$ and $\int_0^\infty \tilde{\mathcal{V}}_+^{\alpha_1}(x) \, dx < \gamma_+^{\alpha_1} < 1$, then equation (42) has a positive solution in $L_1(0, +\infty)$.

Therefore there exists factorization (40). Combining (38) and (40) we get (36).

Taking into account (36), equation (29) yields

$$(I - \hat{\mathcal{V}}_+^{\alpha_1})(I + \hat{B}_{\alpha_1})(I - \hat{\mathcal{V}}_+^{\alpha_1})S = 0.$$  

(43)

The solution of equation (43) is reduced to the successive solution of the following coupled operators equations

$$(I - \hat{\mathcal{V}}_+^{\alpha_1})F = 0,$$

(44)
\[
\begin{align*}
(I + \hat{B}_{\alpha_1})\varphi &= F, \\
(I - \hat{V}^{\alpha_1}_{\tau})S &= \varphi.
\end{align*}
\tag{45}
\tag{46}
\]

We consider the following possible cases for the completely continuous operator \(\hat{B}_{\alpha_1}^*\):

1) " \( -1 \)" is the characteristic value for the operator \(\hat{B}_{\alpha_1}^*\).

2) " \( -1 \)" is the characteristic value for the operator \(\hat{B}_{\alpha_1}^*\).

**Case 1:** We consider the homogeneous equation (44). As the solution of (44) we take the trivial solution \(F(x) \equiv 0\). Since " \( -1 \)" is the characteristic value of the operator \(\hat{B}_{\alpha_1}^*\), then equation \(\hat{B}_{\alpha_1}^* \varphi = -\varphi\) has a non-trivial bounded solution (because \(\hat{B}_{\alpha_1}^*\) is the completely continuous operator in space \(M(0, +\infty)\)). Let us turn to the solution of the equation (46) with respect to the function \(S\):

\[
S(x) = \varphi(x) + \int_0^x \nu_{\alpha_1}(x-t)S(t)dt, \quad x \geq 0,
\tag{47}
\]

where \(\varphi \in M(0, +\infty)\).

If \(\gamma_{\alpha_1}^* < 1\), then equation (47) has a unique solution \(S \in M(0, +\infty)\) in space of bounded functions.

**Case 2:** As a solution of equation (44) we take \(F(x) \equiv 1\) (it is possible because \(\gamma_{\alpha_1}^* = 1\)). Since " \( -1 \)" is not a characteristic value of the operator \(\hat{B}_{\alpha_1}^*\), then due to the complete continuity of the operator \(\hat{B}_{\alpha_1}^*\) in the space \(M(0, +\infty)\) we conclude that there exists the bounded inverse operator \((I + \hat{B}_{\alpha_1}^*)^{-1}\) (see [5]). Since \(F \in M(0, +\infty)\), then \(\varphi \in M(0, +\infty)\). We again come to equation (47), which has bounded solution.

Thus, it has been proved that equation (28) possesses a non-trivial bounded solution \(S(x)\). However, the solution may alternate in sign. Below we prove that equation (28) possesses also a non-trivial non-negative monotonically increasing bounded solution \(S^*(x)\). With this purpose, first let us check that

\[
T_{\alpha_1}(x-t) \geq e^{-2\alpha_1 t}T_{\alpha_1}^0(x+t), \quad \forall (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+.
\tag{48}
\]

Really taking into account (7), (8), (12) and (13) we have

\[
T_{\alpha_1}(x-t) = \mu e^{\alpha_1(x-t)} \int_{x-t}^\infty K(y)dy \geq \mu e^{-2\alpha_1 t} e^{\alpha_1(x+t)} \int_{x+t}^\infty K(y)dy = e^{-2\alpha_1 t} T_{\alpha_1}^0(x+t).
\]

Now we introduce the following iterations (successive approximations) for equation (28):

\[
S_{n+1}(x) = \int_0^\infty \{T_{\alpha_1}(x-t) - e^{-2\alpha_1 t}T_{\alpha_1}^0(x+t)\}S_n(t)dt, \quad x \geq 0,
\tag{49}
\]

where \(S(x)\) is the bounded solution of equation (28), constructed by means of the factorization (36).

By the induction, taking into consideration (48), it is easy to check that

\[
i) \quad S_n(x) \downarrow \text{ with respect to } n, \quad ii) \quad S_n(x) \geq |S(x)|, \quad n = 0, 1, 2, \ldots.
\tag{50}
\]

Therefore the sequence of functions \(\{S_n(x)\}_{n=0}^\infty\) has the pointwise limit \(\lim_{n \to \infty} S_n(x) = S^*(x) \leq \sup_{x \geq 0} |S(x)|\). Moreover, in accordance with B. Levi’s theorem (see [6]), this limit satisfies the following inequalities

\[
|S(x)| \leq S^*(x) \leq \sup_{x \geq 0} |S(x)|, \quad x \in \mathbb{R}^+.
\tag{51}
\]

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We show that $S^*(x) \uparrow$ while $x$. With this aim, we rewrite the iterations (49) in the following form
\begin{equation}
S_{n+1}(x) = \int_{-\infty}^{x} T_{n}(\tau) S_{n}(x - \tau) d\tau - \int_{0}^{\infty} e^{-2\alpha_{1}T_{0}^{\alpha_{1}}(x + t)} S_{n}(t) dt,
\end{equation}
\begin{equation}
S_{0}(x) = \sup_{x \geq 0} |S(x)|, \quad n = 0, 1, 2, \ldots.
\end{equation}
Taking into consideration (15), (52), and using the induction, it is easy to prove that
\begin{equation}
S_{n}(x) \uparrow \text{ with respect to } x, \quad n = 0, 1, 2, \ldots.
\end{equation}
Therefore $S^*(x) \uparrow$ with respect to $x$. Thus the following theorem holds.

**Theorem 1.** Let the conditions (17), (16) be fulfilled and $\nu(T_{n}) < 0$. Then equation (28) possesses a non-negative non-trivial monotonically increasing bounded solution $S^*(x)$.

In the following paragraph, using theorem 1, we will prove the existence theorem for the problem (1)–(2) in space the $W^{1}_{1}(0, +\infty)$.

**3. Solvability of problem (1)–(2)**

The following theorem is true.

**Theorem 2.** Let the conditions (3)–(5) be fulfilled. Assume that there exists $\eta > 0$, such that
\begin{enumerate}
  \item $h(t, z) \uparrow$ with respect to $z$ for each fixed $t \in \mathbb{R}^{+}$ on the interval $[0, \eta e^{-\alpha_{1}(\mu)t}]$, where $\alpha_{1}(\mu)$ is uniquely defined by relation (17) while $\nu(T_{n}) < 0$,
  \item $h(t, z)$ satisfies Caratheodory’s condition on the interval $\mathbb{R}^{+} \times [0, \eta]$ with respect to the arguments $z$, i.e. for each fixed $z \in [0, \eta]$ the function $h(t, z)$ is measurable with respect to $t \in \mathbb{R}^{+}$ and it is continuous with respect to $z$ on the interval $[0, \eta]$ for almost all $t \in \mathbb{R}^{+}$,
  \item $h(t, z) \geq z$, $t \geq 0$, $z \in [0, \eta e^{-\alpha_{1}(\mu)t}]$,
  \item $h(t, \eta e^{-\alpha_{1}(\mu)t}) = \eta e^{-\alpha_{1}(\mu)t}$.
\end{enumerate}
Then the problem (1)–(2) has a non-trivial non-negative bounded solution of the following structure
\begin{equation}
f(x) = e^{-\alpha_{1}(\mu)t} \psi(x),
\end{equation}
in the space $W^{1}_{1}(\mathbb{R}^{+})$ where $\psi \in W^{1}_{\infty}(\mathbb{R}^{+}) \equiv \{ \varphi : \varphi^{(j)} \in L_{\infty}(0, +\infty), j = 0, 1 \},$
\begin{equation}
0 \leq \psi(x) \leq \eta, \quad \psi(x) \equiv 0, \quad x \in \mathbb{R}^{+}.
\end{equation}

**Proof.** Consider the following successive approximations for the equation (26):
\begin{equation}
\psi_{n+1}(\tau) = e^{\alpha_{1}(\mu)T} \int_{0}^{\infty} \{ T(\tau - t) - T_{0}(\tau + t) \} h(t, e^{-\alpha_{1}(\mu)t} \psi_{n}(t)) dt,
\end{equation}
\begin{equation}
\psi_{0}(\tau) \equiv \eta, \quad n = 0, 1, 2, \ldots, \quad \tau \geq 0.
\end{equation}
By the induction, we can easily prove that
\begin{equation}
\psi_{n}(\tau) \downarrow \text{ with respect to } n.
\end{equation}
Indeed,
\begin{equation}
\psi_{1}(\tau) = e^{\alpha_{1}(\mu)T} \int_{0}^{\infty} \{ T(\tau - t) - T_{0}(\tau + t) \} h(t, e^{-\alpha_{1}(\mu)t} \eta) dt \leq \eta \int_{0}^{\infty} T_{\alpha_{1}}(\tau - t) dt \leq \eta.
\end{equation}
since \( \int_{-\infty}^{+\infty} T_n(z)dz = 1 \),

\[
\psi_1(\tau) \geq e^{\alpha_1(\mu)\tau} \int_{0}^{\infty} \{T(\tau - t) - T_0(\tau + t)\}h(t,0)dt = 0.
\]

We assume that \( 0 \leq \psi_n(\tau) \leq \psi_{n-1}(\tau) \) for some \( n \in \mathbb{N} \). Then, due to condition \( j_1 \), equation (56) implies

\[
\psi_{n+1}(\tau) \geq 0 \text{ and } \psi_{n+1}(\tau) \leq e^{\alpha_1(\mu)\tau} \int_{0}^{\infty} \{T(\tau - t) - T_0(\tau + t)\}h(t, e^{-\alpha_1(\mu)t} \psi_{n-1}(t))dt = \psi_n(\tau).
\]

Now we prove that

\[
\psi_n(\tau) \geq \frac{S^*(\tau)}{\sup_{t \geq 0} S^*(t)} \eta, \quad \tau \geq 0, \quad n = 0, 1, 2, \ldots \quad (58)
\]

If \( n = 0 \) then inequality (58) is obviously fulfilled. Let (58) holds true for some \( n \in \mathbb{N} \). Then due to conditions \( j_1 \), equation (56) yields

\[
\psi_{n+1}(\tau) \geq e^{\alpha_1(\mu)\tau} \int_{0}^{\infty} \{T(\tau - t) - T_0(\tau + t)\}h\left(t, \eta e^{-\alpha_1(\mu)t} \frac{S^*(t)}{\sup_{t \geq 0} S^*(t)} \right) dt \geq \\
\geq \frac{\eta}{\sup_{t \geq 0} S^*(t)} \int_{0}^{\infty} \{T_{n+1}(\tau - t) - e^{-2\alpha_1(\mu)t} T_{n+1}(\tau + t)\}S^*(t)dt = \frac{S^*(\tau)}{\sup_{t \geq 0} S^*(t)} \eta,
\]

where \( \alpha_1 \equiv \alpha_1(\mu) \).

Thus, the sequence of functions \( \{\psi_n(\tau)\}_{n=0}^{\infty} \) has a pointwise limit \( \lim_{n \to \infty} \psi_n(\tau) = \psi(\tau) \) while \( n \to \infty \); besides \( \psi(\tau) \) satisfies the following chain of the inequalities

\[
\frac{\eta |S(\tau)|}{\sup_{t \geq 0} S^*(t)} \leq \eta \frac{S^*(\tau)}{\sup_{t \geq 0} S^*(t)} \leq \psi(\tau) \leq \eta, \quad \tau \geq 0.
\] (59)

Taking into the consideration B. Levi’s theorem and condition \( j_2 \) we conclude that the limit function \( \psi(\tau) \) satisfies equation (26). From formula (27) we obtain the following inequalities:

\[
\frac{\eta e^{-\alpha_1(\mu)x} S^*(x)}{\sup_{t \geq 0} S^*(t)} \leq f(x) \leq \eta e^{-\alpha_1(\mu)x}, \quad x \geq 0.
\] (60)

Now let us check that \( \psi \in W^1_\infty(0, +\infty) \). In fact, \( H(x) \equiv \frac{dT}{dx}, \quad \check{H}(x) \equiv \frac{dT_0}{dx} \) are continuous and integrable functions on the sets \((-\infty; +\infty)\) and \((0, +\infty)\) respectively (because of functions \( T \) and \( T_0 \) are given by (7) and (8), and \( K \) is the exponential function of the type (3)). Besides, the integrals

\[
e^{\alpha_1(\mu)x} \int_{0}^{\infty} H(x - t)h(t, e^{-\alpha_1(\mu)t} \psi(t))dt,
\]

\[
e^{\alpha_1(\mu)x} \int_{0}^{\infty} \check{H}(x + t)h(t, e^{-\alpha_1(\mu)t} \psi(t))dt,
\]

\[
e^{\alpha_1(\mu)x} \int_{0}^{\infty} T(x - t)h(t, e^{-\alpha_1(\mu)t} \psi(t))dt,
\]

\[
e^{\alpha_1(\mu)x} \int_{0}^{\infty} T_0(x + t)h(t, e^{-\alpha_1(\mu)t} \psi(t))dt
\]

uniformly (bounded) converge, and $\psi \in M(0, +\infty)$. Hence, due to the theorems on the differentiation of the parameter dependent integral (see [7]), we conclude that $\psi' \in M(0, +\infty)$. Therefore the theorem has been proved.

**Remark 1.** It should be noted that in particular case where $h(t, z) = z$ (it satisfies conditions $j_1 - j_4$) the solution of problem (1)-(2) reduces to the solution of equation (28). But since the solution of equation (28) is bounded function in the case where $\nu(T_\alpha) < 0$ only (in other cases, using results of [4], it can be proved that non-trivial solutions are unbounded). On the other hand, we are looking for bounded solutions to equation (26). Therefore we have to assume that $\nu(T_\alpha) < 0$.

**Remark 2.** It is also interesting to note that if $h(t, z)$ satisfies conditions $j_1 - j_4$, and
1) $h(t, 0) \neq 0$, $t \in \mathbb{R}^+$,
2) there exists number $L \in (0, 1)$, such that

$$|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2|, \quad t \in \mathbb{R}^+, \quad z_1, z_2 \in [0, \eta e^{-\alpha_1(\mu)t}],$$

then it is easy to prove the uniqueness of the solution to problem (1)-(2) in $W_{1,1}^1(\mathbb{R}^+)$. An example of the function $h(t, z)$ satisfying all the assumptions above is the following:

$$h(t, z) = z - \frac{\eta e^{-\alpha_1(\mu)t}}{2} + \eta^2 e^{-2\alpha_1(\mu)t} z + \eta e^{-\alpha_1(\mu)t}.$$

Here the number $L$ can be chosen as follows: $L = \frac{3}{4}$.

At the end of paper we give a number of examples for the function $h(t, z)$:

a) $h(t, z) = \sqrt{\eta e^{-\alpha_1(\mu)t}}z$,

b) $h(t, z) = z + \frac{\eta e^{-\alpha_1(\mu)t}}{\pi} \sin^2 \left(\frac{\pi z}{\eta e^{-\alpha_1(\mu)t}}\right)$,

c) $h(t, z) = \sqrt{\eta e^{-\alpha_1(\mu)t} z e^{\alpha_1(\mu)t} \pi^{-1}}$.

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**References**


О разрешимости одного класса граничных задач нелинейных интегро-дifferentialных уравнений в кинетической теории плазмы

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Работа посвящена исследованию одного класса нелинейных интегро-дифференциальных уравнений с некомпактным оператором Гаммерштейна в полуплоскости. Рассматриваемый класс уравнений имеет прямое приложение в кинетической теории плазмы. Комбинация методов специальной факторизации с теорией построения инвариантных конусных отрезков для нелинейных операторов позволила доказать существование решений рассматриваемых уравнений в пространствах Соболева.

Ключевые слова: факторизация, ядро, монотонность, итерация, условие Каратеодри, пространство Соболева.