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Simple Groups and Sylow Subgroups

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Firstly a problem to characterize Sylow 2-subgroups (of small order $\leqslant 2^{10}$) of finite simple groups is proposed. Next some (possibly necessery) reduction steps are discussde. The latter half of these notes is devoted to finite groups having extra-spacial p-groups of order p^3 as Sylow subgroups.

Keywords: simple group, Sylow subgroups.

Dedicaded to the memory of Professor Vladimir P. Shunkov

1. Classification of finite simple groups

It is a common knowledge that the completion of the classification of all finite simple groups was first 'announced' sometime around 1980, although there were some results not yet printed or announced officially. These 'gaps' were filled in due course except for the classification of quasi-thin groups. (A finite group G is said to be quasi-thin if for every nonidentity 2-subgroup P, the normalizer $N_G(P)$ does not contain an abelian p-subgroup of rank more than 2 with p being an odd prime.) From the mid 1990's to early in the 2000's, Aschbacher and Smith made enormous (and courageous) efforts to put an end to the classification of all finite simple groups by redoing the determination of all quasi-thin groups almost from scratch. Their work was published in 2004 [1]. This work marked the official end of the complete classification of all finite simple groups which had been sought after by group theorists for nearly 150 years or even longer.

A review of the classification [The classification of finite simple groups] was published by Aschbacher, Lyons, Smith and Solomon in 2011 [2]. This book is given the subtitle [Groups of characteristic 2 type], since the book deals mostly with the groups of characteristic 2 type and hence the authors regard their book as the 'Volume 2' of Gorenstein's book [The classification of finite simple groups. Vol. 1. Plenum Press, New York, 1983. Groups of noncharacteristic 2 type]. Their publication earned the 2012 L. P. Steele Prize from the American Mathematical Society.

Independent from the books mentioned above, Lyons, and Solomon (with late Gorenstein) have been devoting efforts in revising the existing classification of finite simple groups since early in 1980's. Up to ten volumes have been planned and the first six volumes have already been published [3] as of 2013.

At this point, one could ask naively: **Is there another way for classifying all simple groups of finite order ?**

This is a question, which would fall not only upon the current group theorists, but surely also upon some future group theorists. In fact, some research in this direction has already been undertaken. Currently, the most active area of mathematics in this direction is the research on the fusion systems. The fusion system was not necessary created to redo the classification of

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finite simple groups, but as its application, the theory has proved useful in that aim also [4]. A little more (but not much more) on the fusion systems will be mentioned later in this notes.

2. Sylow subgroups

Let G be a finite group and $n = |G|$, the order of G. Let

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_s^{e_s}$$

be the factorization of the order n into the product of powers of distinct primes $p_1, p_2, \ldots, p_s$.

Sylow(1872). For each i, G possesses a subgroup (called a Sylow p_i-subgroup) of order $p_i^{e_i}$ and all Sylow p_i-subgroups are conjugate in G.

Without any question, this theorem due to Sylow is the most fundamental result obtained for finite groups. The proof is also easy. Firstly, show that a finite abelian group whose order is divisible by a prime p contains an element of order p. Given this fact, an easy induction, though using the class equation of the group crucially, on the order of groups involved will prove the theorem of Sylow. Here the class equation of a group G is defined to be:

$$|G| = |C_1| + |C_2| + \cdots + |C_s|,$$

where $\{C_i, 1 \leq i \leq s\}$ are the conjugacy classes of G. It has been said that the notion of conjugation was first brought into group theory by Sylow. Induction on the order of groups is used in the proof and so, in general, we do not know the Sylow p-subgroups explicitly. We know only the existence of them.
Example. The symmetric group S_n of degree n. Let

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_s p^s, \quad 0 \leqslant a_i < p$$

be the expansion of n written to the base p. Then

$$S_n \sim_p (S_p)^{a_1} \times (S_{p^2})^{a_2} \cdots \times (S_{p^s})^{a_s},$$

where the notation $\sim_p$ denotes that the Sylow p-subgroups of the both sides are isomorphic. We know also that the Sylow p-subgroup of the symmetric group S_{p^i} is isomorphic to the $(i-1)$-fold wreath product $Z_p \wr Z_p \wr \cdots \wr Z_p$. By this description, the Sylow p-subgroup of S_n is in general the direct product of smaller p-group. In particular, it is not indecomposable, unless n itself is a power of the prime p.
Question. Can we characterize the Sylow subgroups of finite simple groups? Or, a slightly weaker question: What can we say about the structure of Sylow subgroups of a simple group?

If the prime p in question is odd, then we cannot say much. Some may even say that practically nothing can be said if p is odd. On the other hand, if $p = 2$, we know many useful results. For example, the Sylow 2-subgroup of a non-abelian simple group cannot be cyclic, generalized quaternion, $Z_4 \times Z_4$, to name just a few. Note that here we are asking a stand-alone structure theorem on Sylow p-subgroups, but not a result on fusion, or on p-local subgroups.

Let us here raise a more specific question. There are 49,487,365,422 isomorphism types of groups of order 2^{10} [5]. But if we make use of the result of the classification of all finite simple groups, we know that only 11 of them can and will appear as Sylow 2-subgroups of finite simple groups.

Definition. *A p-group P is said to be realizable if P is isomorphic to a Sylow p-subgroup of a (nonabelian) finite simple group.*

PROBLEM. Characterize 11 (known) realizable groups of order 2^{10} among 49,487,365,422 groups.
$\{L_2(2^{10}), L_2(q), L_3(q), G_2(17), S_4(17), L_4(9), M_{24}, Co_3, A_{14}, U_5(2), Sz(32)\}$
is the list of 11 simple groups possessing Sylow 2-subgroups of order 2^{10} (with a suitable choice of q, a dihedral or a semi-dihedral group of order 2^{10} are the relevant 2-groups there). If the number 49,487,365,422 appears intimidating, then there are smaller numbers. We know there are 2,328, 56,092, 10,494,213 groups of order $2^7, 2^8$ and 2^9 respectively ([6–8][19] and there are 7, 7, and 12 realizable groups respectively.)

As for even smaller groups of order $\leqslant 2^6$, there are 267 groups altogether [9][14], and 18 realizables. Those small 2-groups and their occurrences in finite simple groups as Sylow 2-subgroups were investigated by the author in the mid 1960's by hand. The difficult cases to eliminate from the list of realizable 2-groups of order $\leqslant 2^6$ were all decomposable groups such as

$$Z_2 \times Z_2 \times D_8, D_8 \times D_8, \ldots.$$

Those decomposable groups were eliminated in 1970's by the following papers and by others.

Theorem (Gorenstein and Harada [10]). *$D_{2^m} \times D_{2^n} (2 \leqslant m, 3 \leqslant n)$ cannot be a Sylow 2-subgroup of a finite simple group. (D_{2^t} is a dihedral group of order 2^t.)*

Theorem (Goldschmidt [11]). *If $S = S_1 \times S_2$ is a Sylow 2-subgroup of a finite group and suppose $S \cap (S_i)^G \subset S_i (i = 1, 2)$. Then G is not simple. ($(S_i)^G$ is the set of all conjugates of S_i in G. If the condition $S \cap (S_i)^G \subset S_i (i = 1, 2)$ holds, then S_i is said to be strongly closed in S with respect to G.)*

Remark. In the paper of Gorenstein and Harada, each direct factor of a given Sylow 2-subgroup is indecomposable (unless $m = 2$), while in the paper of Goldschmidt, each direct factor S_i is not assumed to be indecomposable.

3. Decomposable and indecomposable Sylow subgroups

We have already shown how the Sylow p-subgroups of the symmetric groups would look like. It says, in particular, the Sylow p-subgroups of S_n is decomposable unless n is a power of the prime p. Since the index $[S_n : A_n] = 2$, the Sylow p-subgroups of the alternating group A_n are isomorphic to those of S_n, if p is an odd prime. On the other hand, if $p = 2$, the situation is very much different.

In fact,

Theorem (([12]). *If T is a Sylow 2-subgroup of the alternating group A_n, then*
(i) if $n = 4, 5,$ or $n = 4m + 2, 4m + 3$ with $4m \neq 2^r$, then T is deomposable; and,
(ii) if $n = 4m, 4m + 1, m \geqslant 2,$ or $n = 4m + 2, 4m + 3$ with $4m = 2^r$, then T is indecomposable.

Remark. We have $A_{4m+2} \sim_2 S_{4m} (\sim_2 A_{4m+3})$. To see it, simply replace all transpositions of a Sylow 2-subgroup of S_{4m} by (12)·(transposition).

Remark. $A_{14} \sim_2 S_4 \times S_8 \sim_2 A_6 \times A_{10}$ is the smallest alternating group for which its Sylow 2-subgroups are decomposable.

From the remark mentioned above, one would suspect that if a Sylow 2-subgroup of a (known) simple group is decomposable, then each indecomposable component is also realizable (with respect to a smaller simple group). In fact one can prove it in general.

Theorem (Harada and Lang [12]). *Let S be a Sylow 2-subgroup of a (known) finite simple group and let $S = S_1 \times S_2 \times \cdots \times S_k$ be the direct product and each component S_i is indecomposable. Then each S_i is also a Sylow 2-subgroup of some (known) simple group.*

4. Reduction of the problem

First Reduction. We may assume that the 2-group S in question is indecomposable.

A supportive evidence to this reduction is the observation of Harada and Lang [12] made on all known simple groups. But the initial PROBLEM is raised in order to redo the classification of all simple groups of finite order. If so, this reduction step must be proved. The prototype of the proof will be Gorenstein and Harada [10] and also Goldschmidt [11]. However, it is best to accept this reduction step as it is. If we find a good characterization of indecomposable realizable 2-groups, then we will, hopefully, be able to find a good formulation of this reduction step.

Second Reduction. We may assume that the 2-group S in question is of nilpotent class at least 3.

We have the following classical results. If the number of group of order p^n is denoted by $p^{a(n,p)}$, then

$$a(n,p) = n^3(2/27 + O(n^{-1/3})).$$

which is due to to G. Higman [13], C. Sims [14]. On the other hand, the number of special group of order p^n is $p^{b(n,p)}$ with

$$b(n,p) = n^3(2/27 + O(n^{-1})),$$

(G. Higman [13].) If a p-group P satisfies $Z(P) = \Phi(P) = [P,P] =$ elementary abelian, then P is said to be special. These results show that as n tends to infinity, then almost all 2-groups are of class 2. In fact, we observe that

2^7, 41%
2^8, 56%
2^9, 84%

are of class 2. For groups of order 2^{10}, perhaps, more than 95% of them will be of class 2 (personal communication, Eick, 2000.)

In order ot support the second reduction theoretically, we have the following results of Gorenstein and Gilman [15], Gomi [16].

Theorem (Gorenstein and Gilman, Gomi)**.** *Let G be a simple group having a Sylow 2-subgroup of nilpotenmt class at most 2. Then G is of a known type.*

These results were obtained in the mid 1970's. If a modern technique is used, perhaps a smoother and shorter argument will prove an equivalent result. This is the mathematical reason for the second reduction step.

Third Reduction. We may assume that the 2-group S in question is of the lowest type.

Sylow 2-subgroups of the simple group $L_2(q)$ is elementary abelian if q is even and dihedral if q is odd. Classification-wise, investigating one even case and one odd case will suffice. In fact, except for a few small cases of linear groups, considering two cases with $q = 2, 3$ will in general be sufficient. Thus, a Sylow 2-subgroup of a simple group G will be called of the *lowest type* if G is a sporadic simple group or G is a linear group defined on the field of two or three elements.

At this moment, we have no more reductions. Summarizing, let us define that a 2-group S is of *basic type* if S is indecomposable, of nilpotent class at least 3 and of the lowest type.

Remark. If one applies the formula of Higman, Sims for the number of 2-groups of order 2^n very naively, one would estimate that there would be about 2^{1500} groups of order 2^{46}. But by the classification of all finite simple groups, we know that only three of them are realizable and of basic type, *i.e.* $Monster, L_{20}(3)$, and $O_{26}^-(3)$.

Strategy

Step 1. Write out generators and relations for the 2-group in question. Use involutions whenever possible. If done so, then fusion arguments will be easier to perform and also easier to view the obtained results.

Step 2. Force the fusion of involutions (or elements of higher orders if necessary) using;
(1). Z^*-theorem [17],
(2). Transfer theorems.

Most 2-groups will be eliminated by this Step 2.

Step 3. For the remaining 2-groups, one can use, and perhaps it is necessary to use, Goldschmidt's theorem [18] on groups having a strongly closed abelian 2-subgroup.

There will be some remaining groups to be eliminated. If so, then we will have to formulate necessary results and prove them. Keep in mind that only 11 out of some 50 billions of 2-groups of order 2^{10} will survive from all criteria (known or unknown.)

5. Sylow subgroups for odd primes

No corresponding Z^*-theorems for an odd prime is known. In particular, if $a \neq 1$ is an element of a Sylow p-subgroup P (p odd) of a finite group G, then we do not know whether or not a is conjugate to another element of P in G under a reasonable assumption of simplicity of G. Glauberman's Z^*-theorem for $p = 2$ has been one of the powerful results in analyzing the structure of Sylow 2-subgroups of simple groups. But for an odd p, the corresponding theorem is totally missing. If we use the classification of all finite simple groups, then we know that there are some parallel results for $p = 2$ and p odd, such as a Z^*-theorem for $P \cong Z_p \times Z_p$. But those desired results have been far from being proved (without the classification.) We note also that a parallel result sometime may be false.

As for stand-alone transfer theorems for odd primes, we know the following:

Theorem (Yoshida [19]). *Assume that the wreath product $Z_p \wr Z_p$ is not involved in a Sylow p-subgroup P, p odd, of a finite group G. Then, if p divides $|N_G(P)/N_G(P)'|$ then p divides $|G/G'|$ also. Here the quotation mark $'$ indicates the commutator subgroup.*

Let us here list up a few desired results for a Sylow p-subgroup P of a finite (simple) group G for an odd prime p. Note that desired results are not necessarily true even for known simple groups. Therefore, 'desired' means : Prove it or determine all counterexamples.

(Desired Result No.1). There is no (conjugacy) isolated elements of order p in P with respect to G, *i.e.* Z^*-theorem for an odd prime p.

(Desired Result No.2). There is no strongly closed subgroups of order p in P with respect to G (a slight extension of the Z^*-theorem.)

(Desired Result No.3). If there is a strongly closed abelian subgroup in P with respect to G, then the group G in question is classifiable.

Note that the ordering of the strength of these desired results is obviously: No.1 $<$ No.2 $<$ No.3.

Let us investigate, as a test, the structure of a finite group G with an explicitly prescribed Sylow p-subgroup P with an odd p. Most results below has already been obtained in Ruiz and Viruel [20]. But, the author of this notes was not aware of their result until sometime later. Ruiz and Viruel's result is based on the axioms and languages of fusion systems, while in this notes, we only use the standard group theory. As in [20], we investigate only the possible fusion patterns and some p local subgroups, but make no attempts to determine the corresponding (global) group structure. To determine the group structure, there will be unsurmountable obstacles at this moment, unless the classification of finite simple groups is used.

Test Case: P is isomorphic to an extra-special group of order p^3 (the exponent of P is not assumed to be p.)

There are two isomorphism types of such an extra-special group P of order p^3. These two types are often denoted symbolically by $p_{\pm}^{1+2}$, where p_{+}^{1+2} is of exponent p and p_{-}^{1+2} is of exponent p^2. Now let G be a finite group having a Sylow p-subgroup P.

Case 1. $P \cong p_{-}^{1+2}$.
Suppose $P = \langle a, b \mid a^p = 1, b^{p^2} = 1, [a,b]^p = 1, [a,b,a] = [a,b,b] = 1\rangle$. P contains only one elementary abelian subgroup A of order p^2. That is $A = \langle a, z\rangle$ where $z = [a,b] \in Z(P)$. Under the automorphism group $Aut(P)$ of P, the element a has $p^2 - p$ images while b has $p^3 - p^2$ images. The elements a and b can be mapped to those elements independently and so we have $|Aut(P)| = p^3(p-1)^2$. We know the inner automorphism group $|Inn(P)| = p^2$ and $|Out(P) = p(p-1)^2$. We have $Out(P) \subset GL_2(p)$ and $Out(P) \cap SL_2(p)$ centralizes z. Therefore, if $N_G(P)/C_G(P)P$ is not embedded in $SL_2(p)$, then z will not be an isolated element.

Next assume the stronger Desired Result No.2. Namely, assume that there is no conjugation isolated subgroups of order p. Obviously, then $\langle z\rangle$ is conjugate in G to another subgroup $\langle y\rangle$ of P in G. Since all elements of order p is contained in $A = \langle a, z\rangle$, we have $y \in A$. By a simple application of Sylow's theorem (or using conjugation family), we see that $\langle z\rangle$ is conjugate to $\langle y\rangle$ in $N_G(A)$. Let $\overline{N} = N_G(A)/C_G(A) \subset GL_2(p)$. Since $\langle z\rangle$ is conjugate to $\langle y\rangle$ in $N_G(A)$, $\overline{N}$ does not have a normal Sylow p-subgroup. This implies that $\overline{N} \supset SL_2(p)$. Considering the normalizer of the matrix -I (minus of the (2 x 2) identity matrix), we conclude that there exists an element of order p outside of A, which is against the structure of $P \cong p_{-}^{1+2}$. So $\langle z\rangle$ is always isolated. Therefore, if the Desired Result No.2 is assumed with $P \cong p_{-}^{1+2}$, then we reach a contradiction.
Conclusion. If $P \cong p_{-}^{1+2}$ is a Sylow p-subgroup of a finite group G, then the center $Z(P)$ of P is always conjugation isolated. That is $Z(P)^G \cap P \subset Z(P)$ always.

Case 2. $P \cong p_{+}^{1+2}$.
In this case, $P = \langle a, b \mid a^p = b^p = 1, [a,b]^p = 1, [a,b,a] = [a,b,b] = 1\rangle$ and P is of exponent p. The automorphism group $Aut(P)$ of P is isomorphic to $Z_p \times Z_p \cdot GL_2(p)$ with $Inn(P) \cong Z_p \times Z_p$. We assume that the Desired Result No. 2 and that G is not a counter-example. Mimicking the argument for the case $P \cong p_{-}^{1+2}$ and using Alperin's theorem of conjugation family, we conclude that there exists $A \cong Z_p \times Z_p$ such that $\overline{N} = N_G(A)/C_G(A) \supset SL_2(p)$. In particular, all nonidentity elements of A are conjugate under $N_G(A)$. The Desired Result No.2 takes us this much but no farther.

So assume the Desired Result No.3 and that G is not a counter-example. In particular, A mentioned above is not a strongly closed subgroup of P. This forces that there is a second subgroup $B \cong Z_p \times Z_p$ such that $\overline{N_G(B)} = N_G(B)/C_G(B) \supset SL_2(p)$. Looking at the list of known simple groups, we see that this possibility actually occurs in $L_3(p)$ and $U_3(p)$. In fact, for these two series of simple groups, there exist exactly two subgroups A, B of order p^2 in P such that the normalizer of each of them involves $SL_2(p)$. Those two subgroups A, B are not arbitrary. In fact, B is the unique subgroup (other than A itself) of P of order p^2 normalized by a cyclic subgroup of order $p-1$ of $SL_2(p) \subset \overline{N_G(A)}$ (strictly speaking B is the inverse image of a subgroup of order p of $\overline{N_G(A)}$ in P.)

Next we consider the cases where there are at least three subgroups isomorphic to $Z_p \times Z_p$ in P which involves $SL_2(p)$ in their normalizers. We first state a well known result of L.E. Dickson [21] below. More than half a century before Dickson's result, E. Galois claimed that the group $PSL_2(p)$ possesses a subgroup of index $\leqslant p$ only when $p = 3, 5, 7,$ and 11. Dickson completely determined all subgroups of $PSL_2(q)$ where q is a power of a prime, and he also determined how those subgroups extend in $PGL_2(q)$. But for simplicity we list only maximal subgroups of $PSL_2(p)$, p an odd prime, and their extensions in $PGL_2(p)$.

Theorem (Dickson [21]). *Maximal subgroups of $PSL_2(p)$ with p odd (and how they extend in $PGL_2(p)$) are as follows.*

(1). semi-direct product of order $\dfrac{p(p-1)}{2}$, *($p(p-1)$ in $PGL_2(p)$,)*

(2). dihedral group of order $p-1$, ($2(p-1)$ in $PGL_2(p)$,)
(3). dihedral group of order $p+1$, ($2(p+1)$ in $PGL_2(p)$,)
(4). for $p = \pm 1 \mod 8$, two conjugacy classes of the symmetric group S_4 of degree 4, (one class in $PGL_2(p)$,)
(5). for $p = \pm 3 \mod 8$, the alternating group A_4 of degree 4, (S_4 in $PGL_2(p)$,)
(6). for $p = \pm 1 \mod 10$, two conjugacy classes of the alternating group A_5 of degree 5, (one class in $PGL_2(p)$.)

Remark. If $p = 5$, then $PSL_2(5) \cong A_5$ and $PGL_2(5) \cong S_5$.

Next we give a small lemma concerning an extension of $Z_p \times Z_p$ by $SL_2(p)$.

Lemma. *Let N be a nontrivial extension of (a normal subgroup) $A \cong Z_p \times Z_p$ by $SL_2(p)$. Then:*
(1). N splits over A, and so N is a semi-direct product of $A \cong Z_p \times Z_p$ by $SL_2(p)$,
(2). if P is a Sylow p-subgroup of N, then $P \cong p_+^{1+2}$,
(3). P possesses $p+1$ subgroups isomorphic to $Z_p \times Z_p$. Let Ω be the set of all such $p+1$ subgroups,
(4). if C is a cyclic subgroup of $SL_2(p)$ of order $p-1$, then the faithful action C^Ω of C on Ω is a cyclic group of oder $\dfrac{p-1}{3}$ if 3 divides $p-1$, and cyclic of order $p-1$ otherwise,
(5). let $r = |C^\Omega|$. Then the orbit lengths of C^Ω on Ω are $\{1,1,r,r,r\}$ or $\{1,1,r\}$ depending on $r = \dfrac{p-1}{3}$ or $r = p-1$. The first orbit of length 1 is A. The second orbit of length 1 is $B = \langle z, b\rangle$ where $b \in SL_2(p)$. Therefore, B is uniquely determined in P if A is chosen.

Proof. The proof is an easy exercise (using matrices.)

Theorem. *Suppose there are at least three subgroups isomorphic to $Z_p \times Z_p$ in P such that the normalizer of each of them involves $SL_2(p)$. Then $p = 3, 5, 7, 11$ or 13.*

Proof. Let
$$\Omega = \{A \cong Z_p \times Z_p \mid A \subset P\}$$
Our Sylow p-subgroup $P \cong p_+^{1+2}$ contains exactly $p+1$ subgroups isomorphic to $Z_p \times Z_p$ and so $|\Omega| = p+1$. The normalizer $N_G(P)$ of P acts on Ω and let us denote by $N_G(P)^\Omega$ the faithful action of $N_G(P)$ on Ω. We have $N_G(P)^\Omega \subset PGL_2(p)$. Now let $B \in \Omega$ such that $N_G(B)$ involves $SL_2(p)$. Then,
$$\overline{N} = N_G(B)/O_{p'}(N_G(B)) \supset (Z_p \times Z_p) \cdot SL_2(p).$$
Now choose $R \subset N_G(B) \cap N_G(P)$ such that $\overline{R}$ is a cyclic subgroup of $\overline{N}$ of order $p-1$. R acts on Ω. As seen in the statement (4) of the previous lemma, we conclude that the faithful action $R^\Omega (\subset N_G(P)^\Omega)$ is of order $p-1$ if $p-1$ is not divisible by 3, and is of order $\dfrac{p-1}{3}$ if $p-1$ is divisible by 3.

Suppose $p \geqslant 17$. Then $\dfrac{p-1}{3} \geqslant 6$. Therefore, R^Ω is a cyclic group of order at least 6 in $N_G(P)^\Omega \subset PGL_2(p)$. We note that p does not divide the order of $N_G(P)^\Omega$, since P is a Sylow p-subgroup of G. Now, from Dickson's list of subgroups, only subgroups of $PGL_2(p)$ that contain a cyclic subgroup of order at least 6 are S_5, cyclic, or dihedral groups. The first possibility does not occur in our case and in the latter two possibilities, there exists only one cyclic subgroup of order at least 6. But in our case, there should be more than one cyclic group of order at least 6, since we assumed that there are at least three subgroups of P isomorphic to $Z_p \times Z_p$ each of whose normalizer involves $SL_2(p)$. This contradiction proves the theorem.

The last theorem would suggest that if $p \geq 17$, then we should be lead (though not yet proved) towards the linear groups $L_3(p)$ and $U_3(p)$. But there are varieties of groups if $p \leq 13$. There are, indeed, surprising varieties of groups as shown below: List taken from [20] (see [22]

also,) but not all possibilities of their lists are shown here.
$p = 3$, Tits' group;
$p = 5$, Thompson's group;
$p = 7$, O'Nan's group;
$p = 11$, Janko's fourth group J_4;
$p = 13$, Monster group.

In order to get the reader interested in this kind of work, let us investigate the following special subproblem.
Problem. Suppose a finite group G possesses a Sylow p- subgroup $P \cong p_+^{1+2}$ for an odd prime p. Moreover, assume that all nonidentity elements of P are conjugate in G. What can we say about G or p-local subgroups.

Again we note that the most work below is contained in [20], where the work is presented with the language of fusion systems. We will write here in the language of 'standard' group theory.

Let G be such a group and P be one of its Sylow p-subgroups. By our assumption, if $1 \neq z \in Z(P)$, then z is conjugate in G to every nonidentity element of P. Choose an arbitrary element $a \in P - Z(P)$ and put $A = \langle z, a \rangle$. Using Alperin's conjugation faimily, we see that a is conjugate to z in $N_G(A)$. This in turns implies that $\overline{N} = N_G(A)/C_G(A)$ contains a subgroup isomorphic to $SL_2(p)$.

Let $M = N_G(P) \cap N_G(A)$ and M^Ω be the faithful action of M on Ω. We know that M^Ω contains a cyclic subgroup R^Ω of order $p-1$ or $\dfrac{p-1}{3}$. Let us again denote $r = p-1$ if $p-1$ is not divisibe by 3, and $r = \dfrac{p-1}{3}$ otherwise. As seen above, R^Ω acts on Ω and orbit lengths are $1, 1, r$, or $1, 1, r, r, r$ This is true with any subgroup of P isomorphic to $Z_p \times Z_p$, and so M^Ω contains $\dfrac{p+1}{2}$ such cyclic subgroups of order r. By the preceding theorem, we already know that $p \leq 13$, since there are $p+1$ subgroups of order p^2 each of whose normailizer involves $SL_2(p)$. Let us deal with the remaining primes one by one.

Case $p = 13$. In this case, $r = 4$ and M^Ω must contain at least $\dfrac{13+1}{2} = 7$ cyclic subgroups of order 4. Dickson's list shows this is impossible.

Case $p = 11$. In this case, $r = 10$. We need at least $\dfrac{11+1}{2} = 6$ cyclic subgroups of order 10. This is not possible by Dickson's result.

Case $p = 7$. In this case, $r = 2$ and we need at least $\dfrac{7+1}{2} = 4$ cyclic subgroups of order 2 in $M^\Omega \subset PGL_2(7)$. These subgroups of order 2 fixes two elements of Ω, and so they are not contained in $PSL_2(7)$. We have three possible subgroups for $N_G(P)^\Omega$, *i.e.* $N_G(P)^\Omega \cong D_8, D_{16}$, or D_{12}. These three configurations have not been eliminated locally but there is no corresponding (known) simple groups. In the language of fusion systems, they are called *exotic* fusion systems (see [20].)

Case $p = 3, 5$. For these cases, there are known examples. Namely, if $p = 3$, then Tits' group has the property required, and if $p = 5$, then Thompson's groups have the required property.

It is a very interesting problem to characterize Tits' group or Thompson's group by this property. But no essential progresses appears having been obtained yet.

6. Notes on fusion systems

To learn the theory of fusion systems (for the first time,) one may wish to read Markus Linckelmann's article [23], and continue on reading two books reviewed by him.

Bob Oliver [4] has recently determined all possible structures of 2-groups of sectional rank at most 4 under the assumptions that the corresponding fusion systems are *reduced* and *indecomposable*. This is in some sense a 'revision' of the result of Gorenstein and Harada [24]. The structures of simple groups themselves are not determined in the Oliver's work.

As mentioned a few times in the previous section, Ruiz and Viruel [20] investigated finite groups having an extra-special Sylow p-subgroup of order p^3 of exponent p. They have found three exotic fusion systems for $p = 7$, which have so far not been eliminated unless the classification of all finite simple groups is used. These three cases (*i.e.* $N_G(P)^{\Omega} \cong D_8, D_{16}$, or D_{12}) were touched on in this notes before.

Modular representation theory on finite groups having an extra-special Sylow p-subgroup of order p^3 of exponent p have been studied. For example see [22] and its extensive references in it. The exotic fusion systems occurring for $p = 7$ appear not have been investigated from modular representation theoretic view points in [22].

References

[1] M.Aschbacher, S.D.Smith, The classification of quasithin groups. I, II, Amer. Math. Soc., Mathematical Surveys and Monographs, 111–112, 2004.

[2] M.Aschabcher, R.Lyons, S.D.Smith, R.Solomon, The classification of finite simple groups; Groups of characteristic 2 type, Amer. Math. Soc., Mathematical Surveys and Monographs, 172, 2011.

[3] D.Gorenstein, R.Lyons, R.Solomon, The classification of finite simple groups, Numbers 1,2,3,4,5, and 6, Amer. Math. Soc., Mathematical Surveys and Monographs, 40, no. 1, 1994.

[4] B.Oliver, Reduced fusion systems over 2-groups of sectional rank at most four, preprint, 2013.

[5] H.U.Besche, B.Eick, E.A.O'Brien, The groups of order at most 2000, *Electron. Res. Announc. Amer. Math. Soc.*, **7**(2001), 1–4.

[6] B.Eick, E.A.O'Brien, Enumerating p-groups, *J. Austral. Math. Soc.*, **7**(1999), 192–205.

[7] E.A.O'Brien, The groups of order 256, *J. Algebra*, **143**(1991), 219–235.

[8] R.James, M.F.Newman, E.A.O'Brien, The groups of order 128, *J. Algebra*, **129**(1990), 136–158.

[9] M.Hall Jr., J.K.Senior, The Groups of order $2^6, (n \leqslant 6)$, Macmillan, New York, 1964.

[10] D.Gorenstein, K.Harada, Finite groups whose Sylow 2-subgroups are the direct product of two dihedral groups, *Ann. Math.*, **95**(1972), 1–54.

[11] D.M.Goldschmidt, Strongly closed 2-subgroups of finite simple groups, *Ann. Math.*, **102**(1975), 475–489.

[12] K.Harada, Mong Lung Lang, Indecomposable Sylow 2-subgroups of simple groups, *Acta Appl. Math.*, (2005), 161–194.

[13] G.Higman, Enumerating p-groups. I, *Proc. London Math. Soc.*, **10**(1960), no. 3, 24–30; Enumerating p-groups. II, 566–582.

[14] C.C.Sims, Enumerating p-groups, *Proc. London Math. Soc.*, **15** (1966), 151–156.

[15] R.Gilman, D.Gorenstein, Finite groups with Sylow 2-subgroups of class two. I, II, *Trans. Amer. Math. Soc.*, **207**(1975), 1–101, 103–126.

[16] K.Gomi, Finite groups all of whose non-2-closed 2-local subgroups have Sylow 2-subgroups of class 2, *J. Algebra*, **35**(1975), 214–223.

[17] G.Glauberman, Central elements in core-free groups, *J. Algebra*, **4**(1966).

[18] D.M.Golschmidt, 2-fusion in finite groups, *Ann. Math.*, **99**(1974), 70–117.

[19] Tomoyuki Yoshida, Character-theoretic transfer, *J. Algebra*, **52**(1978), 1–38.

[20] A.Ruiz, A.Viruel, The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p, *Math. Zeit.*, (2004), 45–65.

[21] L.E.Dickson, Linear groups with an exposition of the Galois field theory, Dover Publications, Inc., New York, 1958.

[22] Ryo Narasaki, Katsuhiro Uno, Isometries and extra special Sylow groups of order p^3, *Jour. Alg.*, **322**(2009), no. 6, 2027–2068.

[23] M.Linckelmann, Bull. Amer. Math. Soc., **50**(2013), 523-526. Review on fusion systems in algebra and topology, by M. Aschbacher, R. Kessar and B. Oliver, London Math. Soc., Leture Notes, Vol. 391, Cambridge Univ. Press., (201), and The theory of fusion systems an algebraic approach, by David A. Craven,

[24] D.Gorenstein, K.Harada, Finite groups whose 2-subgroups are generated by at most 4 elements, *Mem. Amer. Math. Soc.*, **147**(1974), 1–464.

Простые группы и силовские подгруппы

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Рассматривается проблема характеризации силовских 2-подгрупп (небольшого порядка $\leqslant 2^{10}$) конечных простых групп. Описываются некоторые (возможно необходимые) шаги по ее редукции. Оставшаяся часть статьи посвящена конечным группам, силовскими подгруппами порядка p^3 в которых являются экстраспециальные p-подгруппы.

Ключевые слова: простые группы, силовские подгруппы.