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The Joint Motion of Two Binary Mixtures in a Flat Layer

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The invariant solution of the equations of thermodiffusional motion is investigated. This solution describes the motion of two immiscible incompressible binary mixtures with a common flat interface under the action of pressure gradient and thermocapillary forces. The stationary flow of such system is found. If the pressure gradient in one of the mixtures tends to zero sufficiently fast, then the motion of mixtures is slowed down by the viscous friction. On the other hand, if there exists a finite limit of pressure gradient when time tends to infinity, then the solution tends to the stationary state.

Keywords: flat layer, thermodiffusional motion, invariant solution.

1. Problem Statement

Consider the motion of two immiscible incompressible binary mixtures with a common interface. Suppose that Ω_j $(j = 1, 2)$ are the domains occupied by the fluids with interface Γ, $\mathbf{u}_j(\mathbf{x}, t)$ and $p_j(\mathbf{x}, t)$ are the velocity vectors and pressures, respectively, and $\theta_j(\mathbf{x}, t)$ and $c_j(\mathbf{x}, t)$ are the deviations of temperatures and concentrations from their average values. The equations of thermodiffusion and motion *in the absence of external forces* ($\mathbf{g} = 0$) have the form [1]

$$\begin{aligned} &\frac{d\mathbf{u}_j}{dt} + \frac{1}{\rho_j}\nabla p_j = \nu_j \Delta \mathbf{u}_j; \quad \frac{d\theta_j}{dt} = \chi_j \Delta\theta_j; \\ &\frac{dc_j}{dt} = d_j \Delta c_j + \alpha_j d_j \Delta\theta_j; \quad \operatorname{div} \mathbf{u}_j = 0, \end{aligned} \tag{1.1}$$

where ρ_j is the average density, ν_j is the kinematic viscosity, χ_j is the thermal diffusivity, d_j is the diffusion coefficient, α_j is the thermal diffusion coefficient, and $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$.

Suppose that the coefficient of surface tension σ on the interface depends on the temperature and concentration, $\sigma = \sigma(\theta, c)$. For many mixtures, the linear law provides a good approximation of this dependence:

$$\sigma(\theta, c) = \sigma_0 - æ_1(\theta - \theta_0) - æ_2(c - c_0), \tag{1.2}$$

where $æ_1 > 0$ is the temperature coefficient and $æ_2$ is the concentration coefficient (usually $æ_2 < 0$ since the surface tension increases with concentration). Let us now formulate the conditions on the interface Γ.

1. Equality of velocities:

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \mathbf{x} \in \Gamma. \tag{1.3}$$

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2. Kinematic condition:

$$\mathbf{u} \cdot \mathbf{n} = V_{\mathbf{n}}, \quad \mathbf{x} \in \Gamma. \tag{1.4}$$

This condition follows from the assumption that Γ is a moving material surface. Here $\mathbf{n}$ is the unit normal vector to Γ directed from Ω_1 to Ω_2, $V_{\mathbf{n}}$ is the velocity of interface displacement in the normal direction, and $\mathbf{u}$ is the velocity vector on Γ, which is the same for both fluids due to (1.3).

3. Dynamic condition:

$$(P_2 - P_1)\mathbf{n} = 2\sigma H\mathbf{n} + \nabla_\Gamma \sigma, \quad \mathbf{x} \in \Gamma. \tag{1.5}$$

This condition expresses the balance of all forces acting on the surface (pressure, friction, surface tension, and thermocapillary forces). Here $P_j = -p_j + 2\rho_j \nu_j D(\mathbf{u}_j)$ are the stress tensors, D is the rate of strain tensor, H is the mean curvature of Γ, and $\nabla_\Gamma = \nabla - (\mathbf{n} \cdot \nabla)\mathbf{n}$ is the surface gradient.

4. Temperature continuity and concentration balance on the interface:

$$\theta_1 = \theta_2, \quad c_1 = \lambda c_2, \quad \mathbf{x} \in \Gamma, \tag{1.6}$$

where λ is the Henry's law constant.

5. The equality of heat fluxes on the interface:

$$k_2 \frac{\partial \theta_2}{\partial n} - k_1 \frac{\partial \theta_2}{\partial n} = 0, \quad \mathbf{x} \in \Gamma, \tag{1.7}$$

where k_j are the thermal conductivities.

6. The equality of mass fluxes through the interface:

$$d_2 \left(\frac{\partial c_2}{\partial n} + \alpha_2 \frac{\partial \theta_2}{\partial n} \right) = d_1 \left(\frac{\partial c_1}{\partial n} + \alpha_1 \frac{\partial \theta_1}{\partial n} \right), \quad \mathbf{x} \in \Gamma. \tag{1.8}$$

The domains Ω_1 and Ω_2 can be in contact not only with each other, but also with rigid walls that will be denoted by Σ_j. On these walls, the no-slip condition should be imposed

$$\mathbf{u}_j = \mathbf{a}_j(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_j, \tag{1.9}$$

where $\mathbf{a}_j(\mathbf{x}, t)$ is the velocity of the wall Σ_j. In addition, we assume that the temperature on Σ_j satisfies the following conditions

$$\theta_j = \theta_{\mathrm{w}}^j(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_j, \tag{1.10}$$

with given functions θ_{w}^j. It means that temperature is imposed on the wall. The condition of absence of mass flux through the walls Σ_j is written as

$$\frac{\partial c_j}{\partial n} + \alpha_j \frac{\partial \theta_j}{\partial n} = 0, \quad \mathbf{x} \in \Sigma_j. \tag{1.11}$$

For completing the problem statement, the initial conditions should be added to relations (1.1)–(1.6):

$$\mathbf{u}_j(\mathbf{x}, 0) = \mathbf{u}_{0j}(\mathbf{x}), \quad \theta_j(\mathbf{x}, 0) = \theta_{0j}(\mathbf{x}), \quad c_j(\mathbf{x}, 0) = c_{0j}(\mathbf{x}), \quad \mathbf{x} \in \Omega_j. \tag{1.12}$$

In what follows, we consider two-dimensional equations of motion for two binary mixtures with a flat interface in the absence of external forces. It can be shown [2] that this system admits a one-parameter subgroup of transformations corresponding to the generator

$$\frac{\partial}{\partial x} + A_j \frac{\partial}{\partial \theta_j} + B_j \frac{\partial}{\partial c_j} + \rho_j f_j(t) \frac{\partial}{\partial p_j},$$

where A_j, B_j are constants and $f_j(t)$ are functions of time. The invariant solution should be sought in the form

$$u_j = u_j(y,t), \quad v_j = v_j(y,t), \quad p_j = \rho_j f_j(t)x + P_j(y,t),$$

$$\theta_j = A_j x + T_j(y,t), \quad c_j = B_j x + K_j(y,t).$$

It follows from the continuity equation that v_j is a function of time only, $v_j = v_j(t)$. Projecting the momentum equations on y axis, we find $\rho_j^{-1} P_{jy} = v_{jt}(t)$. Further we assume that $v_j(t) = 0$ (otherwise the no-slip conditions on the walls are not satisfied). Then the invariant solution is written as

$$\begin{aligned} &u_j = u_j(y,t), \quad v_j = 0, \quad p_j = \rho_j f_j(t)x + P_j(t), \\ &\theta_j = A_j x + T_j(y,t), \quad c_j = B_j x + K_j(y,t). \end{aligned} \tag{1.13}$$

Solution (1.13) can be interpreted as follows. Suppose that on the interface $y = 0$ between two mixtures the surface tension linearly depends on the temperature and concentration: $\sigma(\theta, c) = \sigma_0 - æ_1\theta - æ_2 c$, where $æ_1 > 0$ and $æ_2$ are constants (see (1.2)). Initially, the first and second mixtures are at rest and occupy the layers $-l_1 < y < 0$ and $0 < y < l_2$, respectively. At $t = 0$, the temperature field $\theta_j = A_j x$ and concentration field $c_j = B_j x$ are created instantly in the entire layers. The thermoconcentration effect and pressure gradients $f_j(t)$ induce the motion of mixtures. In this motion, the interface is represented by the plane $y = 0$ and the trajectories are straight lines parallel to x axis. The functions u_j, T_j, K_j can be called *the perturbations of the quiescent state.*

Substituting (1.13) in the governing equations and taking into account the conditions on the interface $y = 0$, we obtain the initial boundary value problem

$$\begin{aligned} &u_{jt} = \nu_j u_{jyy} + \rho_j f_j(t); \quad T_{jt} = \chi_j T_{jyy} - A u_j; \\ &K_{jt} = d_j K_{jyy} + \alpha_j d_j T_{jyy} - B_j u_j \end{aligned} \tag{1.14}$$

at $-l_1 < y < 0$ $(j = 1)$, $0 < y < l_2$ $(j = 2)$;

$$u_1(0,t) = u_2(0,t), \quad T_1(0,t) = T_2(0,t), \quad K_1(0,t) = \lambda K_2(0,t); \tag{1.15}$$

$$k_1 T_{1y}(0,t) = k_2 T_{2y}(0,t); \tag{1.16}$$

$$d_1(K_{1y}(0,t) + \alpha_1 T_{1y}(0,t)) = d_2(K_{2y}(0,t) + \alpha_2 T_{2y}(0,t)); \tag{1.17}$$

$$\rho_2\nu_2 u_{2y}(0,t) - \rho_1\nu_1 u_{1y}(0,t) = -æ_1 A - æ_2 B_1 \equiv H; \tag{1.18}$$

$$u_j(y,0) = 0, \quad T_j(y,0) = 0, \quad K_j(y,0) = 0. \tag{1.19}$$

In the second equation (1.14), $A \equiv A_1 = A_2$ (it follows from the equality of temperatures at $y = 0$). In the boundary condition (1.15), $\lambda = \text{const}$ is the Henry's law constant, so $B_1 = \lambda B_2$. In addition, ν_j, χ_j, d_j, α_j, k_j are positive constants that characterize the physical properties of the mixtures. The above relations should be supplemented by conditions on the rigid walls $y = -l_1$ and $y = l_2$. These are the no-slip condition

$$u_1(-l_1, t) = 0, \quad u_2(l_2, t) = 0, \tag{1.20}$$

condition of absence of temperature perturbations

$$T_1(-l_1, t) = 0, \quad T_2(l_2, t) = 0, \tag{1.21}$$

and condition of absence of diffusive fluxes

$$\left(\frac{\partial K_1}{\partial y} + \alpha_1 \frac{\partial T_1}{\partial y}\right)\Big|_{y=-l_1} = 0, \quad \left(\frac{\partial K_2}{\partial y} + \alpha_2 \frac{\partial T_2}{\partial y}\right)\Big|_{y=l_2} = 0. \tag{1.22}$$

It can be seen that equations (1.14)–(1.22) form three problems for functions (u_1, u_2), (T_1, T_2), and (K_1, K_2). These problems can be solved successively. Since the problem for the velocity field is linear, it can be decomposed into *inhomogeneous problem* with $f_j(t) \neq 0$ and zero boundary condition (1.18) and *homogeneous problem* with $f_j(t) = 0$ and non-zero boundary condition (1.18), i.e. $H \neq 0$.

Remark 1. *Since $p_1 = p_2$ at $y = 0$ for all x, it follows from the dynamic condition on the interface that [1]*

$$\rho_1 f_1(t) = \rho_2 f_2(t), \quad P_1(t) = P_2(t). \tag{1.23}$$

2. Determination of the Velocity Field Under Given Pressure Gradient

Taking into account the above considerations, let us first consider the problem of determining the velocity field *only under instantly imposed pressure gradient in one of the layers*. In this case, we have the following adjoint linear initial boundary value problem ($f(t) \equiv f_1(t)$)

$$u_{1t} = \nu_1 u_{1yy} + f(t), \quad -l_1 < y < 0; \tag{2.1}$$

$$u_1(-l_1, t) = 0; \tag{2.2}$$

$$u_{2t} = \nu_2 u_{2yy} + \frac{\rho_1}{\rho_2} f(t), \quad 0 < y < l_2; \tag{2.3}$$

$$u_2(l_2, t) = 0; \tag{2.4}$$

$$u_1(0, t) = u_2(0, t), \quad \mu_1 u_{1y}(0, t) = \mu_2 u_{2y}(0, t), \quad t \geqslant 0; \tag{2.5}$$

$$u_1(y, 0) = 0, \quad -l_1 < y < 0, \quad u_2(y, 0) = 0, \quad 0 < y < l_2. \tag{2.6}$$

Relations (2.2) and (2.4) represent the no-slip conditions on the fixed rigid walls, while equations (2.5) express the equality of velocities and shear stresses on the interface [5, p. 268]. In addition, $\nu_{1,2} = \mu_{1,2}/\rho_{1,2}$, where $\mu_{1,2}$ are the dynamical viscosities.

Remark 2. *Without loss of generality, we can assume that $P_1(t) = P_2(t) = 0$ in (1.23) since these functions do not influence the motion of mixtures.*

***A priori* estimates.** Let us derive some *a priori* estimates for the solution of problem (2.1)–(2.6). First, we multiply equation (2.1) by $\varrho_1 u_1(y,t)$ (equation (2.3) by $\varrho_2 u_2(y,t)$) and integrate it with respect to y between $-l_1$ and zero (between zero and l_2). Summing up the obtained relations and using boundary conditions (2.2), (2.4), and (2.5), we find

$$\frac{dE(t)}{dt}+\mu_1\int\limits_{-l_1}^{0}u_{1y}^2\,dy+\mu_2\int\limits_{0}^{l_2}u_{2y}^2\,dy=\rho_1 f(t)\Bigl(\int\limits_{-l_1}^{0}u_1\,dy+\int\limits_{0}^{l_2}u_2\,dy\Bigr), \tag{2.7}$$

where

$$E(t)=\frac{1}{2}\,\rho_1\int\limits_{-l_1}^{0}u_1^2(y,t)\,dy+\frac{1}{2}\,\rho_2\int\limits_{0}^{l_2}u_2^2(y,t) \tag{2.8}$$

is the total energy of two layers.

The uniqueness of solution for problem (2.1)–(2.6) follows from (2.7). It can be seen that if $f(t)=0$, then $u_1(y,t)=u_2(y,t)\equiv 0$.

Relation (2.7) allows us to determine the asymptotic behaviour of solution when $t\to\infty$ under some restrictive assumptions on the function $f(t)$. Indeed, owing to conditions (2.2) and (2.4), the Friedrichs inequalities hold for $u_1(y,t)$ and $u_2(y,t)$:

$$\int\limits_{-l_1}^{0}u_1^2(y,t)\,dy\leqslant\frac{\ell_1^2}{2}\int\limits_{-l_1}^{0}u_{1y}^2(y,t)\,dy,\quad\int\limits_{0}^{\ell_2}u_2^2(y,t)\,dy\leqslant\frac{\ell_2^2}{2}\int\limits_{0}^{\ell_2}u_{2y}^2(y,t)\,dy. \tag{2.9}$$

Using inequalities (2.9) and the Cauchy–Bunyakovski–Schwarz inequality, we find from (2.7) (since $\sqrt{a}+\sqrt{b}\leqslant\sqrt{2(a+b)}$, $a\geqslant 0$, $b\geqslant 0$)

$$\frac{dE(t)}{dt}+4\delta E(t)\leqslant 2\delta_1|f(t)|\sqrt{E(t)}\,, \tag{2.10}$$

where $\delta=\min(l_1^{-2}\nu_1,l_2^{-2}\nu_2)$ and $\delta_1=\rho_1\max(\sqrt{l_1/\rho_1}\,,\sqrt{l_2/\rho_2})$. Taking into account that $E(0)=0$, and according to (2.8) and initial conditions (2.6), we obtain from (2.10)

$$E(t)\leqslant\delta_1^2\Bigl(\int\limits_{0}^{t}|f(t)|e^{2\delta t}\,dt\Bigr)^2e^{-4\delta t}. \tag{2.11}$$

Hence, if the integral

$$\int\limits_{0}^{\infty}|f(t)|e^{2\delta t}\,dt\equiv\sqrt{C_1}\;>0, \tag{2.12}$$

converges, then it follows from (2.11) that

$$E(t)\leqslant\delta_1^2C_1e^{-4\delta t} \tag{2.13}$$

for all $t\geqslant 0$. Therefore, L^2-norms of functions $u_1(y,t)$ and $u_2(y,t)$ tend to zero as $t\to\infty$ exponentially and uniformly with respect to $y\in(-l_2,0)$ and $y\in(0,l_2)$ provided that (2.12) is satisfied. To derive the estimate for $|u_j(y,t)|$, it is necessary to estimate the integrals

$$\int\limits_{-l_1}^{0}u_{1y}^2\,dy,\quad\int\limits_{0}^{l_2}u_{2y}^2\,dy.$$

Let $u(y,t)$ be a solution of the equation $u_t = \nu u_{yy} + F(y,t)$, $y \in [a,b]$. Then the following identity holds:

$$\int_0^t \int_a^b (u_t^2 + \nu^2 u_{yy}^2)\, dydt + \nu \int_a^b u_y^2\, dy = 2\nu \int_0^t (u_t u_y)\Big|_a^b dt + \nu \int_a^b u_{0y}^2\, dy + \int_0^t \int_a^b F^2(y,t)\, dydt, \quad (2.14)$$

where $u_0(y) = u(y,0)$. Identity (2.14) follows from the equalities

$$\int_0^t \int_a^b (u_t - \nu u_{yy})^2\, dydt = \int_0^t \int_a^b F^2(y,t)\, dydt, \qquad u_t u_{yy} = \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2).$$

Let us first put $u = u_1$, $a = -l_1$, $b = 0$, $\nu = \nu_1$, $F = f(t)$ in relation (2.14) and multiply it by ρ_1. Then we take $u = u_2$, $a = 0$, $b = l_2$, $\nu = \nu_2$, $F = \varrho_1 \varrho_2^{-1} f(t)$ and multiply the same relation by ρ_2. Summing up the results, we obtain another integral identity for problem (2.1)–(2.6):

$$\begin{gathered}\rho_1 \int_0^t \int_{-l_1}^0 (u_{1t}^2 + \nu_1^2 u_{1yy}^2)\, dydt + \rho_2 \int_0^t \int_0^{l_2} (u_{2t}^2 + \nu_2^2 u_{2yy}^2)\, dydt + \\ + \mu_1 \int_{-l_1}^0 u_{1y}^2\, dy + \mu_2 \int_0^{l_2} u_{2y}^2\, dy = \rho_1 (l_1 + l_2) \int_0^t f^2(t)\, dt.\end{gathered} \quad (2.15)$$

In the derivation of (2.15), initial conditions (2.2), (2.4), and (2.5) and boundary conditions (2.6) were taken into account. Consequently, for all $t \geqslant 0$

$$\int_{-l_1}^0 u_{1y}^2\, dy \leqslant \frac{E_1(t)}{\mu_1}, \qquad \int_0^{l_2} u_{2y}^2\, dy \leqslant \frac{E_1(t)}{\mu_2}, \quad (2.16)$$

where $E_1(t)$ is the right-hand side of (2.15). Therefore, if

$$\int_0^\infty f^2(t)\, dt \equiv C_2 > 0 \quad (2.17)$$

in addition to (2.12), then the following uniform estimates with respect to y ($y \in (-l_1, 0)$ and $y \in (0, l_2)$) hold:

$$|u_j(y,t)| \leqslant \left(2\delta_1 \sqrt{\frac{2C_1 C_3}{\mu_j \rho_j}}\right)^{1/2} e^{-\delta t}, \quad (2.18)$$

where $C_3 = \rho_1 (l_1 + l_2) C_2$, $j = 1, 2$. These estimates are obtained with the help of

$$u_1^2(y,t) = 2\int_{-l_1}^y u_1(y,t) u_{1y}(y,t)\, dy, \qquad u_2^2(y,t) = -2\int_y^{l_2} u_2(y,t) u_{2y}(y,t)\, dy,$$

and with the help of inequalities (2.7), (2.16), (2.17), and the Cauchy–Bunyakovski–Schwarz inequality.

Remark 3. *It can be shown that if condition* (2.12) *is satisfied, then relation* (2.17) *also holds true.*

We have proved

Theorem 1. *The solution of problem* (2.1)–(2.6) *tends to zero as* $t \to \infty$ *subject to condition* (2.12)*. The rate of convergence satisfies estimates* (2.18) *that are uniform in the intervals* $(-l_1, 0)$ *and* $(0, l_2)$*.*

In other words, if the pressure gradient in one of the mixtures tends to zero sufficiently fast, then the motion of mixtures is slowed down by the viscous friction according to inequalities (2.18).

Solution in Laplace representation. To obtain more detailed information on the behaviour of $u_j(y,t)$, let us apply the Laplace transform to problem (2.1)–(2.6):

$$\tilde{u}_j(y,p) = \int\limits_0^\infty e^{-pt} u_j(y,t)\,dt \quad (j=1,2) \tag{2.19}$$

(the conditions for the applicability of formula (2.19) can be found, for example, in [6, p. 494]). As a result, we obtain a boundary-value problem for representations $\tilde{u}_j(y,p)$:

$$\tilde{u}_1'' - \frac{p}{\nu_1}\,\tilde{u}_1 = -\frac{\tilde{f}(p)}{\nu_1} \quad (-l_1 < y < 0); \tag{2.20}$$

$$\tilde{u}_1(-l_1, p) = 0; \tag{2.21}$$

$$\tilde{u}_2'' - \frac{p}{\nu_2}\,\tilde{u}_2 = -\frac{\varrho_1}{\varrho_2\nu_2}\,\tilde{f}(p) \quad (0 < y < l_2); \tag{2.22}$$

$$\tilde{u}_2(l_2, p) = 0; \tag{2.23}$$

$$\tilde{u}_1(0,p) = \tilde{u}_2(0,p); \tag{2.24}$$

$$\mu_1\tilde{u}_1'(0,p) = \mu_2\tilde{u}_2'(0,p), \tag{2.25}$$

where the prime denotes differentiation with respect to y.

After some calculations, we obtain from (2.20)–(2.25)

$$\begin{gathered}\tilde{u}_1(y,p) = -\frac{\tilde{f}(p)}{pW(p)}\left\{\left[\rho - (\rho-1)\,\mathrm{ch}\sqrt{\frac{p}{\nu_2}}\,l_2\right]\mathrm{sh}\sqrt{\frac{p}{\nu_1}}\,(y+l_1) - \right.\\ \left. -\left(\mathrm{sh}\sqrt{\frac{p}{\nu_1}}\,y + \mathrm{sh}\sqrt{\frac{p}{\nu_1}}\,l_1\right)\mathrm{ch}\sqrt{\frac{p}{\nu_2}}\,l_2 + \frac{\mu}{\sqrt{\nu}}\left(\mathrm{ch}\sqrt{\frac{p}{\nu_1}}\,y - \mathrm{ch}\sqrt{\frac{p}{\nu_1}}\,l_1\right)\mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\right\};\end{gathered} \tag{2.26}$$

$$\begin{gathered}\tilde{u}_2(y,p) = -\frac{\tilde{f}(p)}{pW(p)}\left\{\frac{\mu}{\sqrt{\nu}}\left[1 + (\rho-1)\mathrm{ch}\sqrt{\frac{p}{\nu_1}}\,l_1\right]\mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,(l_2-y) + \right.\\ \left. + \frac{\mu}{\sqrt{\nu}}\,\rho\left(\mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,y - \mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\right)\mathrm{ch}\sqrt{\frac{p}{\nu_1}}\,l_1 + \rho\left(\mathrm{ch}\sqrt{\frac{p}{\nu_2}}\,y - \mathrm{ch}\sqrt{\frac{p}{\nu_2}}\,l_2\right)\mathrm{sh}\sqrt{\frac{p}{\nu_1}}\,l_1\right\}.\end{gathered} \tag{2.27}$$

Here $\tilde{f}(p)$ is the representation of $f(t)$, $\rho = \rho_1/\rho_2$, and

$$W(p) = \mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\,\mathrm{ch}\sqrt{\frac{p}{\nu_1}}\,l_1\left(\frac{\mu}{\sqrt{\nu}} + \mathrm{cth}\sqrt{\frac{p}{\nu_2}}\,l_2\,\mathrm{th}\sqrt{\frac{p}{\nu_1}}\,l_1\right), \quad \mu = \mu_1/\mu_2, \quad \nu = \nu_1/\nu_2. \tag{2.28}$$

The originals $u_j(y,t)$ $(j=1,2)$ are reconstructed by the formula

$$u_j(y,t) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} e^{pt}\tilde{u}_j(y,p)\,dp. \tag{2.29}$$

Suppose that $\lim\limits_{t\to\infty} f(t) = f_0 = \text{const}$ exists, then $\lim\limits_{p\to 0} p\tilde{f}(p) = f_0$ [6, p. 521]. Of course, in this case the function $f(t)$ does not satisfy condition (2.12). Let us calculate $\lim\limits_{p\to 0} p\tilde{u}_j(y,p)$ according to (2.26) and (2.27). Simple but cumbersome calculations with the use of asymptotic representations $\operatorname{sh} x \sim x + x^3/6$, $\operatorname{ch} x \sim 1 + x^2/2$ as $x \to 0$ show that

$$\lim_{p\to 0} p\tilde{u}_1(y,p) = \frac{l_1^2 f_0}{2\nu_1}\left[-\left(\frac{y}{l_1}\right)^2 + \frac{\mu - l^2}{l(\mu+l)}\left(\frac{y}{l_1}\right) + \frac{\mu(l+1)}{l(\mu+l)}\right] \equiv u_1^0(y); \tag{2.30}$$

$$\lim_{p\to 0} p\tilde{u}_2(y,p) = \frac{l_2^2 f_0 \mu}{2\nu_1}\left[-\left(\frac{y}{l_2}\right)^2 + \frac{\mu - l^2}{\mu+l}\left(\frac{y}{l_2}\right) + \frac{l(l+1)}{\mu+l}\right] \equiv u_2^0(y), \tag{2.31}$$

where the relation $\varrho\nu = \mu$ was employed. It can be easily checked that the right-hand sides of (2.30) and (2.31) represent *the exact stationary solution* of problem (2.1)–(2.6), where $f(t)$ should be replaced by f_0. So, the solution of problem (2.1)–(2.6) approaches the stationary regime $u_1^0(y)$, $u_2^0(y)$ as $t \to \infty$.

Solution for semi-bounded layers. To construct this solution, we consider the case when l_1 and l_2 tend to infinity in formulae (2.26), (2.27). Taking into account that relation (2.28) when l_1, $l_2 \to \infty$ becomes

$$W(p) \sim \left(1 + \frac{\mu}{\sqrt{\nu}}\right)\exp\left(\sqrt{\frac{p}{\nu_1}}\,l_1 + \sqrt{\frac{p}{\nu_2}}\,l_2\right)$$

and denoting the limits of $\tilde{u}_j(y,p,l_1,l_2)$ by $\widetilde{U}_j(y,p)$, after some calculations we find

$$\widetilde{U}_1(y,p) = \frac{\tilde{f}(p)}{p}\left[1 + \frac{\sqrt{\nu}\,(\varrho-1)}{\mu+\sqrt{\nu}}\,\exp\left(\sqrt{\frac{p}{\nu_1}}\,y\right)\right]; \tag{2.32}$$

$$\widetilde{U}_2(y,p) = \frac{\tilde{f}(p)}{p}\left[\varrho - \frac{\mu(\varrho-1)}{\mu+\sqrt{\nu}}\,\exp\left(-\sqrt{\frac{p}{\nu_2}}\,y\right)\right]. \tag{2.33}$$

It can be easily checked that $\widetilde{U}_1$, $\widetilde{U}_2$ satisfy problem (2.20), (2.22), (2.24), (2.25) (we recall that $y<0$ in (2.32) and $y>0$ in (2.33)).

Using the properties of inverse Laplace transform [6, p. 506, p. 510], we reconstruct the originals

$$U_1(y,t) = \int\limits_0^t f(\tau)\left[1 + \frac{\sqrt{\nu}\,(\varrho-1)}{\mu+\sqrt{\nu}}\,\mathrm{Erf}\left(-\frac{y}{2\sqrt{\nu_1(t-\tau)}}\right)\right]d\tau; \tag{2.34}$$

$$U_2(y,t) = \int\limits_0^t f(\tau)\left[\varrho - \frac{\mu(\varrho-1)}{\mu+\sqrt{\nu}}\,\mathrm{Erf}\left(\frac{y}{2\sqrt{\nu_2(t-\tau)}}\right)\right]d\tau, \tag{2.35}$$

where

$$\mathrm{Erf}\, z = 1 - \mathrm{erf}\, z, \quad \mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\int\limits_0^z e^{-z^2}\,dz.$$

Formulae (2.34) and (2.35) provide the solution of problem (2.1), (2.3), (2.5), (2.6) in semi-bounded layers.

Suppose that

$$f(t) = \frac{f_1}{\sqrt{t}} \tag{2.36}$$

with the constant f_1. Then, after some calculations, we find from (2.34) and (2.35) (formulae (2.32) and (2.33) can also be used):

$$U_1(y,t) = 2f_1\sqrt{t}\left\{1 + \frac{\sqrt{\nu}\,(\varrho-1)}{\mu+\sqrt{\nu}}\left[\exp\left(-\frac{\xi_1}{4}\right) + \frac{1}{2}\,\xi_1\int\limits_{-\infty}^{\xi_1}\exp\left(-\frac{\xi^2}{4}\right)d\xi\right]\right\}; \tag{2.37}$$

$$U_2(y,t) = 2f_1\sqrt{t}\left\{\varrho - \frac{\mu(\varrho-1)}{\mu+\sqrt{\nu}}\left[\exp\left(-\frac{\xi_2^2}{4}\right) - \frac{1}{2}\,\xi_2\int\limits_{\xi_2}^{\infty}\exp\left(-\frac{\xi^2}{4}\right)d\xi\right]\right\}, \tag{2.38}$$

where $\xi_j = y/\sqrt{\nu_j t}$ is the similarity variable. In other words, if the pressure gradient is given by (2.36), then the solution of problem (2.1), (2.3), (2.5), (2.6) is self-similar and given by formulae (2.37) and (2.38). It is not surprising since only in this case equations (2.1) and (2.3) are invariant under the group of dilatations $u' = au$, $y' = ay$, $t' = a^2 t$ with parameter a.

From (2.37) and (2.38), we find the asymptotic behaviour of velocities as $t \to \infty$ ($\xi_j \to 0$) at any finite y, $|y| \leqslant M = \mathrm{const}$

$$U_j(y,t) = \frac{f_1(\mu + \varrho\sqrt{\nu})}{\mu+\sqrt{\nu}}\,\sqrt{t}\,[1 + O(1)].$$

On the other hand, when t is fixed and $|y| \to \infty$ ($\xi_1 \to -\infty$, $\xi_2 \to +\infty$), one obtains from (2.37) and (2.38)

$$U_1(y,t) = 2f_1\sqrt{t}\left[1 + O\left(\exp\left(-\xi_1^2/4\right)\right)\right], \quad U_2(y,t) = 2f_1\varrho\sqrt{t}\left[1 + O\left(\exp\left(-\xi_2^2/4\right)\right)\right].$$

In the derivation of these relations, the results of asymptotic behaviour of integrals of the type

$$F(z) = \int\limits_z^{\infty} f(\xi)\exp[-S(\xi)]\,d\xi$$

as $z \to \infty$ were used [7, p. 58].

On determining the pressure gradient. Often the volume flow rate through the layers is specified instead of the pressure gradient:

$$Q_1(t) = \int\limits_{-l_1}^{0} u_1(y,t)\,dy, \quad Q_2(t) = \int\limits_{0}^{l_2} u_2(y,t)\,dy. \tag{2.39}$$

For example, suppose that $(-l_1, 0)$ is the layer of water and $(0, l_2)$ is that of oil. The flow rate of oil $Q_2(t)$ is given. Applying the Laplace transform (2.19) to relations (2.39) and using formulae (2.26), (2.27), we find

$$\widetilde{Q}_1(p) = -\frac{\tilde{f}(p)}{pW(p)}\left\{\sqrt{\frac{\nu_1}{p}}\left(\operatorname{ch}\sqrt{\frac{p}{\nu_1}}\,l_1 - 1\right)\left[\varrho - (\varrho-2)\operatorname{ch}\sqrt{\frac{p}{\nu_2}}\,l_2\right] + \right.$$

$$\left. + \frac{\mu}{\sqrt{\nu}}\sqrt{\frac{\nu_1}{p}}\operatorname{sh}\sqrt{\frac{p}{\nu_1}}\,l_1\operatorname{sh}\sqrt{\frac{p}{\nu_2}}\,l_2 - l_1\left(\operatorname{sh}\sqrt{\frac{p}{\nu_1}}\,l_1\operatorname{ch}\sqrt{\frac{p}{\nu_2}}\,l_2 + \frac{\mu}{\sqrt{\nu}}\operatorname{ch}\sqrt{\frac{p}{\nu_1}}\,l_1\operatorname{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\right)\right\}; \tag{2.40}$$

$$\widetilde{Q}_2(p)=-\frac{\tilde{f}(p)}{pW(p)}\left\{\frac{\mu}{\sqrt{\nu}}\sqrt{\frac{\nu_2}{p}}\left(\operatorname{ch}\sqrt{\frac{p}{\nu_2}}\,l_2-1\right)\left[1+(2\varrho-1)\operatorname{ch}\sqrt{\frac{p}{\nu_1}}\,l_1\right]+\right.$$
$$\left.+\varrho\sqrt{\frac{\nu_2}{p}}\operatorname{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\operatorname{sh}\sqrt{\frac{p}{\nu_1}}\,l_1-\varrho l_2\left(\frac{\mu}{\sqrt{\nu}}\operatorname{sh}\sqrt{\frac{p}{\nu_2}}\,l_2\operatorname{ch}\sqrt{\frac{p}{\nu_1}}\,l_1+\operatorname{ch}\sqrt{\frac{p}{\nu_2}}\,l_2\operatorname{sh}\sqrt{\frac{p}{\nu_1}}\,l_1\right)\right\}. \tag{2.41}$$

One can determine $\tilde{f}(p)$ from (2.41) and reconstruct $f(t)$ according to formula (2.29). The flow rate of the first liquid (water) is determined from (2.40) and (2.29).

It is interesting to calculate the flow rate for stationary flows (2.30) and (2.31). In this case,

$$Q_1^0=\int\limits_{-l_1}^{0}u_1^0(y)\,dy=\frac{f_0l_1^3}{12\nu_1 l(\mu+l)}\,(4\mu l+3\mu+l^2),\quad Q_2^0=\int\limits_{0}^{l_2}u_2^0(y)\,dy=\frac{f_0l_2^3\mu}{12\nu_1(\mu+l)}\,(\mu+4l+3l^2).$$

The ratio between flow rates

$$\frac{Q_2^0}{Q_1^0}=\frac{\mu}{l^2}\,\frac{(\mu+4l+3l^2)}{(4\mu l+3\mu+l^2)}$$

strongly depends on the thickness of the layers. For example, if we take $l=0.25$ ($l_2=4l_1$), then for water and oil with $\mu=0.312$ we find $Q_2^0/Q_1^0\approx 5.71$, while for $l=0.5$ ($l_2=2l_1$) we have $Q_2^0/Q_1^0\approx 2.11$.

3. Determination of Velocity Perturbations Induced by Thermocapillary Forces

In this case, the initial boundary value problem is written as

$$u_{1t}=\nu_1u_{1yy},\quad -l_1<y<0; \tag{3.1}$$
$$u_1(-l_1,t)=0; \tag{3.2}$$
$$u_{2t}=\nu_2u_{2yy},\quad 0<y<l_2; \tag{3.3}$$
$$u_2(l_2,t)=0; \tag{3.4}$$
$$u_1(0,t)=u_2(0,t),\quad \mu_2u_{2y}(0,t)-\mu_1u_{1y}(0,t)=H,\quad t\geqslant 0; \tag{3.5}$$
$$u_1(y,0)=0,\quad -l_1<y<0,\quad u_2(y,0)=0,\quad 0<y<l_2. \tag{3.6}$$

Remark 4. *There is a discontinuity in condition* (3.5) *at the initial moment of time since its left-hand side is zero at* $t=0$ *according to* (3.6) *but* $H\neq 0$.

Problem (3.1)–(3.6) has a stationary solution (Couette flow in layers)

$$u_1^0=a\left(1+\frac{y}{l_1}\right),\quad u_2^0=a\left(1-\frac{y}{l_2}\right), \tag{3.7}$$

where

$$a=-\frac{Hl_1}{\mu_2(\mu+l)},\quad H=-(æ_1A+æ_2B_1),\quad l=\frac{l_1}{l_2}. \tag{3.8}$$

The application of Laplace transform (2.19) to problem (3.1)–(3.6) leads to the boundary-value problem

$$\tilde{u}_1''-\frac{p}{\nu_1}\,\tilde{u}_1=0,\quad -l_1<y<0; \tag{3.9}$$

$$\tilde{u}_1(-l_1, p) = 0; \tag{3.10}$$

$$\tilde{u}_2'' - \frac{p}{\nu_2}\,\tilde{u}_2 = 0, \quad 0 < y < l_2; \tag{3.11}$$

$$\tilde{u}_2(l_2, p) = 0; \tag{3.12}$$

$$\tilde{u}_1(0, p) = \tilde{u}_2(0, p); \tag{3.13}$$

$$\mu_2 \tilde{u}_2'(0, p) - \mu_1 \tilde{u}_1'(0, p) = \frac{H}{p}\,, \tag{3.14}$$

where the prime denotes differentiation with respect to y. The solution of problem (3.9)–(3.14) can be easily obtained

$$\tilde{u}_1(y, p) = -\frac{\sqrt{\nu_2}\,H}{\mu_2\sqrt{p^3}\,W_1(p)\,\mathrm{ch}\sqrt{p\nu_1^{-1}}l_1}\ \mathrm{sh}\sqrt{\frac{p}{\nu_1}}\,(l_1 + y), \quad -l_1 < y < 0; \tag{3.15}$$

$$\tilde{u}_2(y, p) = -\frac{\sqrt{\nu_2}\,H\,\mathrm{th}\sqrt{p\nu_1^{-1}}\,l_1}{\mu_2\sqrt{p^3}\,W_1(p)\,\mathrm{sh}\sqrt{p\nu_2^{-1}}\,l_2}\ \mathrm{sh}\sqrt{\frac{p}{\nu_2}}\,(l_2 - y), \quad 0 < y < l_2, \tag{3.16}$$

where

$$W_1(p) = \frac{\mu}{\sqrt{\nu}} + \mathrm{th}\sqrt{\frac{p}{\nu_1}}\,l_1\,\mathrm{cth}\sqrt{\frac{p}{\nu_2}}\,l_2. \tag{3.17}$$

From (3.15)–(3.17), one can find the limits

$$\lim_{p \to 0} p\tilde{u}_j(y, p) = u_j^0(y)$$

with the functions $u_j^0(y)$ from (3.7) and (3.8) as it should be.

The flow rates are given by

$$Q_1^0 = \int\limits_{-l_1}^{0} u_1^0(y)\,dy = \frac{al_1}{2}\,, \quad Q_2^0 = \int\limits_{0}^{l_2} u_2^0(y)\,dy = \frac{al_2}{2}\,, \tag{3.18}$$

and their ratio is $Q_2^0/Q_1^0 = 1/l$.

***A priori* estimates.** Let us introduce new functions

$$w_j(y, t) = u_j^0(y) - u_j(y, t). \tag{3.19}$$

Then $w_j(y, t)$ satisfy the problem

$$w_{1t} = \nu_1 w_{1yy}, \quad -l_1 < y < 0; \tag{3.20}$$

$$w_{2t} = \nu_2 w_{2yy}, \quad 0 < y < l_2; \tag{3.21}$$

$$w_1(0, t) = w_2(0, t), \quad \mu_2 w_{2y}(0, t) - \mu_1 w_{1y}(0, t) = 0; \tag{3.22}$$

$$w_1(-l_1, t) = 0, \quad w_2(l_2, t) = 0; \tag{3.23}$$

$$w_1(y, 0) = u_1^0(y), \quad w_2(y, 0) = u_2^0(y). \tag{3.24}$$

Note that now the initial conditions are non-zero, and the second boundary condition in (3.22) is satisfied for any $t > 0$ (at $t = 0$, its right-hand side equals to H).

Let us multiply equation (3.20) by $\rho_1 w_1$ and integrate it with respect to y between $-l_1$ and 0:

$$\frac{\partial}{\partial t}\frac{1}{2}\rho_1\int\limits_{-l_1}^{0} w_1^2\,dy = \mu_1 w_1 w_{1y}\Big|_{-l_1}^{0} - \mu_1\int\limits_{-l_1}^{0} w_{1y}^2\,dy.$$

Similarly,

$$\frac{\partial}{\partial t}\frac{1}{2}\rho_2\int\limits_{0}^{l_2} w_2^2\,dy = \mu_2 w_2 w_{2y}\Big|_{0}^{l_2} - \mu_2\int\limits_{0}^{l_2} w_{2y}^2\,dy.$$

Summing up these equalities and using boundary conditions (3.22) and (3.23), we obtain

$$\frac{dE}{dt} + \mu_1\int\limits_{-l_1}^{0} w_{1y}^2\,dy + \mu_2\int\limits_{0}^{l_2} w_{2y}^2\,dy = \begin{cases} 0, & t>0;\\ \dfrac{H^2 l_1}{\mu_2(\mu+l)}, & t=0,\end{cases} \tag{3.25}$$

where the 'kinetic' energy of layers is given by

$$E(t) = \frac{1}{2}\rho_1\int\limits_{-l_1}^{0} w_1^2\,dy + \frac{1}{2}\rho_2\int\limits_{0}^{l_2} w_2^2\,dy. \tag{3.26}$$

The Friedrichs inequalities (2.9) hold for w_j due to boundary conditions (3.23). Then from (3.25) we derive the inequality ($\delta = \min(l_1^{-2}\nu_1, l_1^{-2}\nu_2)$)

$$\frac{dE}{dt} + 4\delta E \leqslant h(t), \tag{3.27}$$

where $h(t)$ is the right-hand side of (3.25). Integration of (3.27) with initial conditions (3.24) leads to

$$E(t) \leqslant E(0)e^{-4\delta t}, \tag{3.28}$$

where

$$E(0) = \frac{1}{2}\rho_1\int\limits_{-l_1}^{0} w_1^2(y,0)\,dy + \frac{1}{2}\rho_2\int\limits_{0}^{l_2} w_2^2(y,0)\,dy = \frac{a^2}{6}(\rho_1 l_1 + \rho_2 l_2). \tag{3.29}$$

due to boundary conditions (3.7) and (3.8).

Remark 5. *In the derivation of relation* (3.28), *the Gronuoll inequality was used [8, p. 183]. It is applicable since* $h(t)$ *is a summable function and integral of it is equal to zero.*

Hence,

$$\int\limits_{-l_1}^{0} w_1^2\,dy \leqslant \frac{2E(0)}{\rho_1}e^{-4\delta t}, \quad \int\limits_{0}^{l_2} w_2^2\,dy \leqslant \frac{2E(0)}{\rho_2}e^{-4\delta t}. \tag{3.30}$$

To estimate the L^2-norms of w_{jy}, we again apply identity (2.14). Then, instead of (2.15) we obtain

$$\begin{gathered}\rho_1\int\limits_0^t\int\limits_{-l_1}^{0}(w_{1t}^2 + \nu_1^2 w_{1yy}^2)\,dydt + \rho_2\int\limits_0^t\int\limits_0^{l_2}(w_{2t}^2 + \nu_2^2 w_{2yy}^2)\,dydt + \mu_1\int\limits_{-l_1}^{0} w_{1y}^2\,dy\\ +\mu_2\int\limits_0^{l_2} w_{2y}^2\,dy = \mu_1\int\limits_{-l_1}^{0}(u_{1y}^0)^2\,dy + \mu_2\int\limits_0^{l_2}(u_{2y}^0)^2\,dy = a^2\left(\frac{\mu_1}{l_1} + \frac{\mu_2}{l_2}\right) \equiv D_1.\end{gathered} \tag{3.31}$$

It follows that

$$\int_{-l_1}^{0} w_{1y}^2\,dy \leqslant \frac{D_1}{\mu_1}\,, \quad \int_{0}^{l_2} w_{2y}^2\,dy \leqslant \frac{D_1}{\mu_2}\,. \tag{3.32}$$

Now from (3.30)–(3.32) and the Cauchy–Bunyakovski–Schwarz inequality, we derive the *a priori* estimates

$$|w_j(y,t)| \leqslant 2\sqrt{\frac{2E(0)D_1}{\rho_j\mu_j}}\,e^{-2\delta t}, \tag{3.33}$$

where $E(0)$ and D_1 are given by formulae (3.29) and (3.31), respectively.

Returning to substitution (3.19), we obtain the following result.

Theorem 2. *The solution of initial boundary value problem* (3.1)–(3.6) *is unique and approaches the stationary state* (3.7) *as* $t \to \infty$. *The rate of convergence is estimated by*

$$|u_j(y,t) - u_j^0(y)| \leqslant 2\sqrt{\frac{2E(0)D_1}{\rho_j\mu_j}}\,e^{-2\delta t} \tag{3.34}$$

with constants $E(0)$ *and* D_1 *from* (3.29) *and* (3.31)*, respectively.*

According to (3.34), the solution of initial boundary value problem (3.1)–(3.6) converges exponentially to the stationary solution.

4. Evolution of Temperature Perturbations

In this case, the initial boundary value problem has the form

$$T_{1t} = \chi_1 T_{1yy} - Au_1, \quad -l_1 < y < 0; \tag{4.1}$$

$$T_1(-l_1,t) = 0; \tag{4.2}$$

$$T_{2t} = \chi_2 T_{2yy} - Au_2, \quad 0 < y < l_2; \tag{4.3}$$

$$T_2(l_2,t) = 0; \tag{4.4}$$

$$T_1(0,t) = T_2(0,t), \quad k_1T_{1y}(0,t) = k_2T_{2y}(0,t); \tag{4.5}$$

$$T_1(y,0) = 0, \quad T_2(y,0) = 0. \tag{4.6}$$

Note that boundary conditions (4.5) are identically satisfied at $t = 0$ as well.

Problem (4.1)–(4.6) exactly coincides with problem (2.1)–(2.6), where one should replace $f(t)$ by $-Au_1(y,t)$, $\rho_1\rho_2^{-1}f(t)$ by $-Au_2(y,t)$, ν_j by χ_j, and μ_j by k_j. Note that $\chi_j = k_j/\rho_jc_{0j}$, where c_{0j} are the specific heats of the mixtures. Let us multiply equation (4.1) by $\rho_1c_{01}T_1$ (equation (4.3) by $\rho_2c_{02}T_2$), integrate it with respect to y between $-l_1$ and 0 (between 0 and l_2), and sum up the results. Similarly to (2.7), we find

$$\frac{dE_2}{dt} + k_1\int_{-l_1}^{0} T_{1y}^2\,dy + k_2\int_{0}^{l_2} T_{2y}^2\,dy = -A\Big[\rho_1c_{01}\int_{-l_1}^{0} u_1T_1\,dy + \rho_2c_{02}\int_{0}^{l_2} u_2T_2\,dy\Big], \tag{4.7}$$

where

$$E_2(t) = \frac{1}{2}\,\rho_1 c_{01}\int\limits_{-l_1}^{0} T_1^2\,dy + \frac{1}{2}\,\rho_2 c_{02}\int\limits_{0}^{l_2} T_2^2\,dy. \tag{4.8}$$

The velocity field induced by the pressure gradient only. In this case, estimate (2.13) holds. It follows that

$$\int\limits_{-l_1}^{0} u_1^2\,dy \leqslant \frac{2\delta_1^2 C_1 e^{-4\delta t}}{\rho_1}\,, \qquad \int\limits_{0}^{l_2} u_2^2\,dy \leqslant \frac{2\delta_1^2 C_1 e^{-4\delta t}}{\rho_2}\,. \tag{4.9}$$

The functions $T_j(y,t)$ satisfy the Friedrichs inequalities (2.9). So, from (4.7) we obtain the inequality similar to (2.10):

$$\frac{dE_2}{dt} + 4\delta_2 E_1(t) \leqslant 2\delta_3\sqrt{E_2(t)}\,e^{-2\delta t},$$

where $\delta_2 = \min(l_1^{-2}\chi_1, l_2^{-2}\chi_2)$ and $\delta_3 = \sqrt{2}\,|A|\delta_1\sqrt{C_1}\max(\sqrt{c_{01}}, \sqrt{c_{02}})$. It follows that

$$E_2(t) \leqslant \begin{cases} \dfrac{\delta_3^2}{4(\delta_2-\delta)^2}\,\left(e^{-2\delta t} - e^{-2\delta_2 t}\right)^2, & \delta_2 \neq \delta; \\ \delta_3^2 t^2 e^{-4\delta_2 t}, & \delta_2 = \delta. \end{cases} \tag{4.10}$$

In the derivation of estimate (4.10), we take into account that $E_2(0) = 0$ due to (4.8) and initial conditions (4.6).

The estimates of integrals

$$\int\limits_{-l_1}^{0} T_{1y}^2\,dy, \qquad \int\limits_{0}^{l_2} T_{2y}^2\,dy$$

are obtained from identity (2.14), where one should replace ν_j by χ_j, u_j by T_j, and F_j by $-Au_j$. By analogy with (2.15) we can obtain the following identity

$$\begin{gathered} \rho_1 c_{01}\int\limits_{0}^{t}\int\limits_{-l_1}^{0}(T_{1t}^2 + \chi_1^2 T_{1yy}^2)\,dydt + \rho_2 c_{02}\int\limits_{0}^{t}\int\limits_{0}^{l_2}(T_{2t}^2 + \chi_2^2 T_{2yy}^2)\,dydt + \\ + k_1\int\limits_{-l_1}^{0} T_{1y}^2\,dy + k_2\int\limits_{0}^{l_2} T_{2y}^2\,dy = A^2\left[\rho_1 c_{01}\int\limits_{0}^{t}\int\limits_{-l_1}^{0} u_1^2\,dydt + \rho_2 c_{02}\int\limits_{0}^{t}\int\limits_{0}^{l_2} u_2^2\,dydt\right]. \end{gathered} \tag{4.11}$$

With the help of inequalities (4.9), it follows from (4.11) that

$$\int\limits_{-l_1}^{0} T_{1y}^2\,dy \leqslant \frac{\delta_4(1-e^{-4\delta t})}{k_1}\,, \qquad \int\limits_{0}^{l_2} T_{2y}^2\,dy \leqslant \frac{\delta_4^2(1-e^{-4\delta t})}{k_2}\,, \tag{4.12}$$

where

$$\delta_4 = \frac{A^2\delta_1^2 C_1}{2\delta}\,(c_{01} + c_{02}).$$

Since

$$T_1^2(y,t) = 2\int\limits_{-l_1}^{y} T_1(y,t)T_{1y}(y,t)\,dy, \qquad T_2^2(y,t) = -2\int\limits_{y}^{l_2} T_2(y,t)T_{2y}(y,t)\,dy,$$

we obtain the following estimates from (4.10), (4.12) and (4.8):

$$T_1^2 \leqslant 2\Big(\int\limits_{-l_1}^{0} T_1^2\,dy\Big)^{1/2}\Big(\int\limits_{-l_1}^{0} T_{1y}^2\,dy\Big)^{1/2} \leqslant 2\sqrt{\frac{2\delta_4 E_2(t)}{k_1\rho_1 c_{01}}},$$

or

$$|T_1(y,t)| \leqslant \left(2\sqrt{\frac{2\delta_4 E_2(t)}{k_1\rho_1 c_{01}}}\right)^{1/2}. \tag{4.13}$$

Similarly,

$$|T_2(y,t)| \leqslant \left(2\sqrt{\frac{2\delta_4 E_2(t)}{k_2\rho_2 c_{02}}}\right)^{1/2}. \tag{4.14}$$

Therefore, in this case *the temperature perturbations decay exponentially with time* (as $e^{-\delta t}$ for $\delta \leqslant \delta_2$ and as $e^{-\delta_2 t}$ for $\delta > \delta_2$).

The application of Laplace transform to (4.1)–(4.6) leads to the following boundary value problem for representations

$$\tilde{T}_1'' - \frac{p}{\chi_1}\tilde{T}_1 = \frac{A\tilde{u}_1(y,p)}{\chi_1}, \quad -l_1 < y < 0; \tag{4.15}$$

$$\tilde{T}_2'' - \frac{p}{\chi_2}\tilde{T}_2 = \frac{A\tilde{u}_2(y,p)}{\chi_2}, \quad 0 < y < l_2; \tag{4.16}$$

$$\tilde{T}_1(0,p) = \tilde{T}_2(0,p), \quad k\tilde{T}_1'(0,p) = \tilde{T}_2'(0,p); \tag{4.17}$$

$$\tilde{T}_1(-l_1,p) = 0; \tag{4.18}$$

$$\tilde{T}_2(l_2,p) = 0, \tag{4.19}$$

where $k = k_1/k_2$ and the prime denotes differentiation with respect to y. The solution of problem (4.15), (4.16) can be written as

$$\tilde{T}_1(y,p) = L_1\,\mathrm{sh}\sqrt{\frac{p}{\chi_1}}\,y + L_2\,\mathrm{ch}\sqrt{\frac{p}{\chi_1}}\,y + \frac{A}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{y}\tilde{u}_1(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{\chi_1}}\,(y-z)\right]dz; \tag{4.20}$$

$$\tilde{T}_2(y,p) = L_3\,\mathrm{sh}\sqrt{\frac{p}{\chi_2}}\,y + L_4\,\mathrm{ch}\sqrt{\frac{p}{\chi_2}}\,y + \frac{A}{\chi_2\sqrt{p\chi_2^{-1}}}\int\limits_{0}^{y}\tilde{u}_2(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{\chi_2}}\,(y-z)\right]dz. \tag{4.21}$$

From (4.17)–(4.19) we have the system of algebraic equations for $L_i(p)$, $i = \overline{1,4}$:

$$L_2 - \frac{A}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\,\mathrm{sh}\sqrt{\frac{p}{\chi_1}}\,z\,dz = L_4,$$

$$k\sqrt{\frac{p}{\chi_1}}\,L_1 + \frac{kA}{\chi_1}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\,\mathrm{ch}\sqrt{\frac{p}{\chi_1}}\,z\,dz = \sqrt{\frac{p}{\chi_2}}\,L_3,$$

$$-\,\mathrm{sh}\sqrt{\frac{p}{\chi_1}}\,L_1 + \mathrm{ch}\sqrt{\frac{p}{\chi_1}}\,l_1 L_2 = 0,$$

$$\operatorname{sh}\sqrt{\frac{p}{\chi_2}}\,l_2L_3+\operatorname{ch}\sqrt{\frac{p}{\chi_2}}\,l_2L_4+\frac{A}{\chi_2\sqrt{p\chi_2^{-1}}}\int\limits_0^{l_2}\tilde{u}_2(z,p)\operatorname{sh}\left[\sqrt{\frac{p}{\chi_2}}\,(l_2-z)\right]dz=0.$$

It follows that

$$L_1=\frac{G_1(p)-G_2(p)}{W_2(p)},\quad L_2=L_1\operatorname{th}\sqrt{\frac{p}{\chi_1}}\,l_1,$$
$$L_3=\frac{k}{\sqrt{\chi}}\,L_1-G_1,\quad L_4=L_2-\frac{A}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\operatorname{sh}\sqrt{\frac{p}{\chi_1}}\,z\,dz, \tag{4.22}$$

where the following notations are used

$$G_1(p)=-\frac{kA}{\chi_1}\sqrt{\frac{\chi_2}{p}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\operatorname{ch}\sqrt{\frac{p}{\chi_1}}\,z\,dz,$$

$$G_2(p)=-\frac{A\operatorname{cth}\sqrt{p\chi_2^{-1}}\,l_2}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\operatorname{sh}\sqrt{\frac{p}{\chi_1}}\,z\,dz+$$
$$+\frac{A}{\chi_2\sqrt{p\chi_2^{-1}}\operatorname{sh}\sqrt{p\chi_2^{-1}}\,l_2}\int\limits_0^{l_2}\tilde{u}_2(z,p)\operatorname{sh}\left[\sqrt{\frac{p}{\chi_2}}\,(l_2-z)\right]dz, \tag{4.23}$$
$$W_2(p)=\frac{k}{\sqrt{\chi}}+\operatorname{th}\sqrt{\frac{p}{\chi_1}}\,l_1\operatorname{cth}\sqrt{\frac{p}{\chi_2}}\,l_2.$$

Let us find the stationary solution of problem (4.1)–(4.5) (boundary conditions (4.6) are not taken into account here). We have the following problem for functions $T_1^0(y)$ and $T_2^0(y)$:

$$T^0_{1yy}=\frac{A}{\chi_1}\,u_1^0(y),\quad -l_1<y<0; \tag{4.24}$$

$$T^0_{2yy}=\frac{A}{\chi_2}\,u_2^0(y),\quad 0<y<l_2; \tag{4.25}$$

$$T_1^0(-l_1)=0,\quad T_2^0(l_2)=0; \tag{4.26}$$

$$T_1^0(0)=T_2^0(0),\quad kT^0_{1y}(0)=T^0_{2y}(0),\quad k=k_1/k_2. \tag{4.27}$$

If we substitute the functions $u_1^0(y)$, $u_2^0(y)$ from (2.30) and (2.31) to the right-hand sides of (4.24)–(4.27), then integration of (4.24)–(4.27) and further simplification lead to

$$T_1^0(y)=\frac{Al_1^2f_0}{2\chi_1\nu_1}\left[-\frac{y^4}{12l_1^2}+\frac{(\mu-l^2)y^3}{6l_1l(\mu+l)}+\frac{\mu(l+1)y^2}{2l(\mu+l)}\right]+a_1y+a_2,$$
$$T_2^0(y)=\frac{Al_2^2f_0\mu}{2\chi_2\nu_1}\left[-\frac{y^4}{12l_2^2}+\frac{(\mu-l^2)y^3}{6l_2(\mu+l)}+\frac{l(l+1)y^2}{2(\mu+l)}\right]+ka_1y+a_2, \tag{4.28}$$

where the constant a_1, a_2 are given by

$$a_1=\frac{Al_1^3f_0}{24\chi_1\nu_1(\mu+l)(k+l)}\left[l^3(5\mu l+4\mu+l^2)-\chi\mu(\mu+4l^2+5l)\right],$$
$$a_2=-\frac{Al_1l_2^3f_0}{24\chi_1\nu_1(\mu+l)(k+l)}\left[kl^2(5\mu l+4\mu+l^2)+\chi\mu(\mu+4l^2+5l)\right]. \tag{4.29}$$

It can be shown that $\lim\limits_{t\to\infty} T_j(y,t) = T_j^0(y)$, i.e. the temperature perturbations in the layers approach the stationary state with time if $\lim\limits_{t\to\infty} f(t) = f_0$. To prove this, it is sufficient to calculate the limits $\lim\limits_{p\to 0} p\tilde{T}_j(y,p)$. As an example, let us consider the case $j=1$. First, we recast expression (4.20) with the help of (4.22)

$$\tilde{T}_1(y,p) = \frac{G_1(p)-G_2(p)}{W_2(p)\,\mathrm{ch}\sqrt{p\chi_1^{-1}}\,l_1}\,\mathrm{sh}\sqrt{\frac{p}{\chi_1}}\,(y+l_1)+ \\ +\frac{A}{\chi_1\sqrt{\chi_1^{-1}p}}\int\limits_{-l_1}^{y}\tilde{u}_1(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{\chi_1}}\,(y-z)\right]dz. \tag{4.30}$$

Second, we substitute $\tilde{u}_j(y,p)$ from (2.26) and (2.27) into (4.22), (4.23), and (4.30) and obtain a cumbersome expression for $\tilde{T}_1(y,p)$, which is not presented here. However, there is an easier way of calculating the limit $\lim\limits_{p\to 0} p\tilde{T}_1(y,p)$ from (4.30) and the limits $\lim\limits_{p\to 0} p\tilde{u}_j(y,p) = u_j^0(y)$ given by formulae (2.30) and (2.31). As $p\to 0$ ($\mathrm{sh}\,x\sim x$, $\mathrm{ch}\,x\sim 1$, $x\to 0$), it follows from (4.23) that

$$W_2(p)\sim\frac{k+l}{\sqrt{\chi}}\,,\quad pG_1(p)\sim-\frac{kA}{\chi_1\sqrt{\chi_2^{-1}p}}\int\limits_{-l_1}^{0}u_1^0(z)\,dz,$$

$$pG_2(p)\sim\frac{A}{\chi_1 l_2\sqrt{p\chi_2^{-1}}}\left[-\int\limits_{-l_1}^{0}u_1^0(z)z\,dz+\chi\int\limits_{0}^{l_2}u_2^0(z)(l_2-z)\,dz\right].$$

The integrals in the right-hand sides can be easily calculated with the help of (2.30) and (2.31):

$$\int\limits_{-l_1}^{0}u_1^0(z)\,dz=\frac{f_0l_1^3}{12\nu_1 l(\mu+l)}\,(4\mu l+3\mu+l^2),$$

$$\int\limits_{-l_1}^{0}u_1^0(z)z\,dz=-\frac{f_0l_1^4}{24\nu_1 l(\mu+l)}\,(3\mu l+2\mu+l^2),$$

$$\int\limits_{0}^{l_2}u_2^0(z)\,dz=\frac{f_0l_2^3\mu}{12\nu_1(\mu+l)}\,(\mu+3l^2+4l).$$

Therefore,

$$\lim_{p\to 0}\frac{pG_1(p)-pG_2(p)}{W_2(p)\,\mathrm{ch}\sqrt{p\chi_1^{-1}}\,l_1}\,\mathrm{sh}\sqrt{\frac{p}{\chi_1}}\,(y+l_1)= \\ =-\frac{Af_0l_2^3[kl^2(8\mu l+6\mu+2l^2)+l^3(3\mu l+2\mu+l^2)-\mu\chi(\mu+4l^2+5l)]}{24\nu_1\chi_1(\mu+l)(k+l)}\,(y+l_1). \tag{4.31}$$

The second term in the right-hand side of (4.30) multiplied by p has the following limit as $p\to 0$

$$\frac{A}{\chi_1}\int\limits_{-l_1}^{y}u_1^0(z)(y-z)\,dz=\frac{Af_0l_1^2}{2\nu_1\chi_1}\left\{-\frac{y^4}{12l_1^2}+\frac{(\mu-l^2)y^3}{6l_1l(\mu+l)}+\frac{\mu(l+1)y^2}{2l(\mu+l)}+\right. \\ \left.+\frac{l_1(8\mu l+6\mu+2l^2)y+l_1^2(3\mu l+2\mu+l^2)}{12l(\mu+l)}\right\}. \tag{4.32}$$

Summing up (4.31) and (4.32) gives precisely formula (4.28) for $T_1^0(y)$. It can be shown similarly that $\lim\limits_{p\to 0} p\tilde{T}_2(y,p) = T_2^0(y)$.

Determination of temperature perturbation induced by thermoconcentration forces. Let us first find the stationary solution of problem (4.1)–(4.5) with the functions $u_1^0(y)$, $u_2^0(y)$ from (3.7) and (3.8) in the right-hand sides of equations (4.1) and (4.3). In this case, the functions $T_j^0(y)$ satisfy the boundary value problem (4.24)–(4.27). The integration gives

$$T_1^0(y) = \frac{aA}{\chi_1}\left(\frac{y^3}{6l_1} + \frac{y^2}{2}\right) + a_1 y + a_2, \quad T_2^0(y) = \frac{aA}{\chi_2}\left(-\frac{y^3}{6l_2} + \frac{y^2}{2}\right) + ka_1 y + a_2, \tag{4.33}$$

$$a_1 = \frac{aAl_2(l^2-\chi)}{3\chi_1(k+l)}, \quad a_2 = -\frac{aAl_1l_2(kl+\chi)}{3\chi_1(k+l)}.$$

Here the functions $T_j^0(y)$ are expressed by third-degree polynomials in y in contrast to (4.28). As in the previous paragraph, in this case it can be shown with the help of (4.20)–(4.23) and (3.15)–(3.17) that $\lim\limits_{p\to 0} p\tilde{T}_j(y,p) = T_j^0(y)$. So, the temperature perturbation approaches the stationary regime with time.

5. Evolution of Concentration Perturbations in the Layers

The initial boundary value problem for concentration perturbations has the form

$$K_{1t} = d_1 K_{1yy} + \frac{\alpha_1 d_1}{\chi_1} T_{1t} + \left(\frac{\alpha_1 d_1 A}{\chi_1} - \lambda B_2\right) u_1; \tag{5.1}$$

$$K_{2t} = d_2 K_{2yy} + \frac{\alpha_2 d_2}{\chi_2} T_{2t} + \left(\frac{\alpha_2 d_2 A}{\chi_2} - B_2\right) u_2; \tag{5.2}$$

$$K_1(0,t) = \lambda K_2(0,t), \quad d(K_{1y}(0,t) + \alpha_1 T_{1y}(0,t)) = K_{2y}(0,t) + \alpha_2 T_{2y}(0,t); \tag{5.3}$$

$$K_{1y}(-l_1,t) + \alpha_1 T_{1y}(-l_1,t) = 0, \quad K_{2y}(l_2,t) + \alpha_2 T_{2y}(l_2,t) = 0; \tag{5.4}$$

$$K_1(y,0) = 0, \quad K_2(y,0) = 0. \tag{5.5}$$

Equations (5.1) and (5.2) are satisfied for $-l_1 < y < 0$ and $0 < y < l_2$, respectively. The term T_{jyy} was replaced from the second equation (1.14). In addition, $B_1 = \lambda B_2$. So, (5.1) and (5.2) are inhomogeneous parabolic equations with known right-hand sides (see sections 2–4). In boundary condition (5.3), $d = d_1/d_2$.

Stationary distribution of concentrations. To find this distribution, we assume that $K_{jt} = 0$ and $T_{jt} = 0$. Then one obtains the following *boundary value problem* instead of (5.1)–(5.4):

$$K_{1yy}^0 = \left(\frac{\lambda B_2}{d_1} - \frac{\alpha_1 A}{\chi_1}\right) u_1^0(y), \quad -l_1 < y < 0; \tag{5.6}$$

$$K_{2yy}^0 = \left(\frac{B_2}{d_2} - \frac{\alpha_2 A}{\chi_2}\right) u_2^0(y), \quad 0 < y < l_2; \tag{5.7}$$

$$K_1^0 = \lambda K_2(0), \quad d(K_{1y}(0) + \alpha_1 T_{1y}^0(0)) = K_{2y}^0(0) + \alpha_2 T_{2y}^0(0); \tag{5.8}$$

$$K_{1y}^0(-l_1) + \alpha_1 T_{1y}^0(-l_1) = 0, \quad K_{2y}^0(l_2) + \alpha_2 T_{2y}^0(l_2) = 0, \tag{5.9}$$

where the functions $u_j^0(y)$, $T_j^0(y)$ are given by formulae (2.30), (2.31) [(3.7), (3.8))], (4.28), (4.29) [(4.33)]. The choice of particular functions depends on the factor that induces the motion of mixtures, i. e., the pressure gradient or thermoconcentration forces. In the former case, we substitute $u_1^0(y)$ from (2.30) into (5.6) and $u_2^0(y)$ from (2.31) into (5.7). With the help of the first condition in (5.8), integration leads to

$$K_1^0(y)=\frac{l_1^2 f_0}{2\nu_1}\left(\frac{\lambda B_2}{d_1}-\frac{\alpha_1 A}{\chi_1}\right)\left[-\frac{y^4}{12l_1^2}+\frac{(\mu-l^2)y^3}{6l_1 l(\mu+l)}+\frac{\mu(l+1)y^2}{2l(\mu+l)}\right]+b_1 y+\lambda b_2,\ -l_1<y<0;$$

$$K_2^0(y)=\frac{l_2^2 f_0\mu}{2\nu_1}\left(\frac{B_2}{d_2}-\frac{\alpha_2 A}{\chi_2}\right)\left[-\frac{y^4}{12l_2^2}+\frac{(\mu-l^2)y^3}{6l_2(\mu+l)}+\frac{l(l+1)y^2}{2(\mu+l)}\right]+b_3 y+b_2,\ 0<y<l_2. \tag{5.10}$$

The constants b_1, b_3 are found from the boundary conditions on the walls (5.9):

$$b_1=-\alpha_1 T_{1y}^0(-l_1)+\frac{l_1^3 f_0}{12\nu_1 l(\mu+l)}\left(\frac{\lambda B_2}{d_1}-\frac{\alpha_1 A}{\chi_1}\right)(4\mu l+3\mu+l^2),$$

$$b_3=-\alpha_2 T_{2y}^0(l_2)-\frac{l_2^3 f_0\mu}{12\nu_1(\mu+l)}\left(\frac{B_2}{d_2}-\frac{\alpha_2 A}{\chi_2}\right)(3l^2+4l+\mu)$$

The second condition (5.8) on the interface provides the following relation

$$db_1-b_3=(k\alpha_2-d\alpha_1)a_1, \tag{5.11}$$

where a_1 is a constant from (4.29). Since it follows from (4.28) that

$$T_{1y}^0(-l_1)=a_1-\frac{l_1^3 A f_0}{12\chi_1\nu_1 l(\mu+l)}(4\mu l+3\mu+l^2),$$

$$T_{2y}^0(l_2)=ka_1+\frac{l_2^3 A f_0\mu}{12\chi_2\nu_1(\mu+l)}(3l^2+4l+\mu),$$

condition (5.11) is satisfied if and only if $B_2=0$. Therefore, *the stationary distribution of concentrations is possible only in the absence of their gradients in the direction of motion at the initial moment of time. When $B_2\neq 0$, the distribution is always non-stationary.*

So, if $B_2=0$, then we have in (5.10)

$$b_1=-\alpha_1 a_1,\quad b_3=-\alpha_2 k a_1. \tag{5.12}$$

The constant b_2 remains arbitrary and without loss of generality it can be assumed to be zero since adding constant concentrations λb_2 and b_2 to K_1^0 and K_2^0, respectively, does not change the problem for $K_j^0(y)$.

In the case when the velocity field is determined from (3.7) and (3.8) and the perturbation of temperatures are found from (4.33), integration of equations (5.6) and (5.7) gives

$$K_1^0(y)=\left(\frac{\lambda B_2}{d_1}-\frac{\alpha_1 A}{\chi_1}\right)a\left(\frac{y^3}{6l_1}+\frac{y^2}{2}\right)+b_1 y+\lambda b_2,$$

$$K_2^0(y)=\left(\frac{B_2}{d_2}-\frac{\alpha_2 A}{\chi_2}\right)a\left(-\frac{y^3}{6l_2}+\frac{y^2}{2}\right)+b_3 y+b_2,\quad 0<y<l_2, \tag{5.13}$$

where one should again put $B_2=0$. The constants b_1 and b_3 are given by (5.12), where a_1 is a constant from (4.33).

So, one should put $B_2 = 0$ in equations (5.1) and (5.2) when studying the behaviour of solution for the problem (5.1)–(5.5) as $t \to \infty$.

Solution in Laplace representation. Let us find the solution of problem (5.1)–(5.5) using the Laplace transform. Taking into account zero initial data for K_j and T_j, we obtain the following boundary problem for representations $\tilde{K}_j(y,p)$:

$$\tilde{K}_1'' - \frac{p}{d_1}\tilde{K}_1 = -\frac{\alpha_1}{\chi_1}p\tilde{T}_1 + \left(\frac{\lambda B_2}{d_1} - \frac{\alpha_1 A}{\chi_1}\right)\tilde{u}_1; \tag{5.14}$$

$$\tilde{K}_2'' - \frac{p}{d_2}\tilde{K}_2 = -\frac{\alpha_2}{\chi_2}p\tilde{T}_2 + \left(\frac{B_2}{d_2} - \frac{\alpha_2 A}{\chi_2}\right)\tilde{u}_2; \tag{5.15}$$

$$\tilde{K}_1(0,p) = \lambda\tilde{K}_2(0,p), \quad d\tilde{K}_1'(0,p) - \tilde{K}_2'(0,p) = \alpha_2\tilde{T}_2'(0,p) - \alpha_1 d\tilde{T}_1'(0,p); \tag{5.16}$$

$$\tilde{K}_1(-l_1,p) = -\alpha_1\tilde{T}_1'(-l_1,p), \quad \tilde{K}_2(l_2,p) = -\alpha_2\tilde{T}_2'(l_2,p). \tag{5.17}$$

Note that the right-hand side of (5.16) is equal to

$$\alpha_2\tilde{T}_2'(0,p) - \alpha_1 d\tilde{T}_1'(0,p) = (k\alpha_2 - \alpha_1 d)\tilde{T}_1'(0,p). \tag{5.18}$$

due to (4.17). The solution of problem (5.14)–(5.17) is written as

$$\tilde{K}_1(y,p) = D_1\,\mathrm{sh}\sqrt{\frac{p}{d_1}}\,y + D_2\,\mathrm{ch}\sqrt{\frac{p}{d_1}}\,y + \sqrt{\frac{d_1}{p}}\int\limits_{-l_1}^{y} h_1(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{d_1}}\,(y-z)\right]dz; \tag{5.19}$$

$$\tilde{K}_2(y,p) = D_3\,\mathrm{sh}\sqrt{\frac{p}{d_2}}\,y + D_4\,\mathrm{ch}\sqrt{\frac{p}{d_2}}\,y + \sqrt{\frac{d_2}{p}}\int\limits_{0}^{y} h_2(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{d_2}}\,(y-z)\right]dz, \tag{5.20}$$

where

$$h_1 = -\frac{\alpha_1}{\chi_1}p\tilde{T}_1 + \left(\frac{\lambda B_2}{d_1} - \frac{\alpha_1 A}{\chi_1}\right)\tilde{u}_1, \quad h_2 = -\frac{\alpha_2}{\chi_2}p\tilde{T}_2 + \left(\frac{B_2}{d_2} - \frac{\alpha_2 A}{\chi_2}\right)\tilde{u}_2. \tag{5.21}$$

After substituting (5.19) and (5.20) into boundary conditions (5.16) and (5.17), we find $D_j(p)$ ($j = 1,2,3,4$) with the help of (5.18)

$$D_1(p) = \frac{\lambda}{W_3(p)}\left[G_3(p) - G_4(p) - \alpha_2\sqrt{\frac{d_2}{p}}\,\frac{\tilde{T}_2'(l_2,p)}{\mathrm{ch}\sqrt{pd_2^{-1}}\,l_2} - \frac{\alpha_1}{\lambda}\sqrt{\frac{d_1}{p}}\,\frac{\mathrm{th}\sqrt{pd_2^{-1}}\,l_2}{\mathrm{sh}\sqrt{pd_1^{-1}}\,l_1}\tilde{T}_2'(-l_1,p)\right],$$

$$D_2(p) = \sqrt{\frac{d_1}{p}}\,\frac{\alpha_1\tilde{T}_1'(-l_1,p)}{\mathrm{sh}\sqrt{pd_1^{-1}}\,l_1} + D_1(p)\,\mathrm{cth}\sqrt{\frac{p}{d_1}}\,l_1,$$

$$D_3(p) = \sqrt{d}\,D_1(p) - G_3(p), \quad d = d_1/d_2,$$

$$D_4(p) = \frac{D_2(p)}{\lambda} - \frac{1}{\lambda}\sqrt{\frac{d_1}{p}}\int\limits_{-l_1}^{0} h_1(z,p)\,\mathrm{sh}\sqrt{\frac{p}{d_1}}\,z\,dz, \tag{5.22}$$

$$G_3(p) = -d\sqrt{\frac{d_2}{p}}\int\limits_{-l_1}^{0} h_1(z,p)\,\mathrm{ch}\sqrt{\frac{p}{d_1}}\,z\,dz + \sqrt{\frac{d_2}{p}}\,(k\alpha_1 - \alpha_1 d)\tilde{T}_1'(0,p),$$

$$G_4(p) = -\frac{1}{\lambda}\sqrt{\frac{d_1}{p}}\ \mathrm{th}\sqrt{\frac{p}{d_2}}\, l_2 \int\limits_{-l_1}^{0} h_1(z,p)\,\mathrm{sh}\sqrt{\frac{p}{d_1}}\, z\, dz +$$

$$+\frac{1}{\mathrm{ch}\sqrt{pd_2^{-1}}\, l_2}\sqrt{\frac{d_2}{p}}\int\limits_0^{l_2} h_2(z,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{d_2}}\,(l_2 - z)\right] dz,$$

$$W_3(p) = \lambda\sqrt{d} + \mathrm{cth}\sqrt{\frac{p}{d_1}}\, l_1\, \mathrm{th}\sqrt{\frac{p}{d_2}}\, l_2.$$

Using formulae (5.19)–(5.22), it can be shown that $\lim\limits_{p\to 0} p\tilde{K}_j(y,p) = K_j^0(y)$ when $B_2 = 0$, where K_j^0 are given by (5.10) or (5.13). It is done in the same way as in section 4.

On *a priori* estimate of concentration perturbations. Let us write equations (5.1) and (5.2) in the form

$$K_{1t} = d_1 K_{1yy} + \alpha_1 d_1 T_{1yy} - \lambda B_2 u_1, \quad -l_1 < y < 0; \tag{5.23}$$

$$K_{2t} = d_2 K_{2yy} + \alpha_2 d_2 T_{2yy} - B_2 u_2, \quad 0 < y < l_2. \tag{5.24}$$

We integrate these equations with respect to y, taking into account the second boundary condition (5.3) and conditions (5.4) and (5.5). As a result,

$$\int\limits_{-l_1}^{0} K_1\, dy + \int\limits_0^{l_2} K_2\, dy = -B_2\left[\lambda\int\limits_0^t\int\limits_{-l_1}^{0} u_1\, dydt + \int\limits_0^t\int\limits_0^{l_2} u_2\, dydt\right].$$

One can only deduce from this relation that

$$\left|\int\limits_{-l_1}^{0} K_1\, dy + \int\limits_0^{l_2} K_2\, dy\right|$$

is bounded for $t \geqslant 0$. In particular, this expression is zero when $B_2 = 0$ (note that $K_j(y,t)$ can have arbitrary signs since they represent the concentration perturbations).

On the other hand, multiplying (5.23) and (5.24) by K_1 and K_2, respectively, and integrating again with respect to y, we obtain the integral identity

$$\begin{gathered}\frac{dE_3}{dt} + d_1\int\limits_{-l_1}^{0} K_{1y}^2\, dy + d_2\int\limits_0^{l_2} K_{2y}^2\, dy = -\alpha_1 d_1\int\limits_{-l_1}^{0} K_{1y}T_{1y}\, dy - \\ -\alpha_2 d_2\int\limits_0^{l_2} K_{2y}T_{2y}\, dy - B_2\left(\lambda\int\limits_{-l_1}^{0} u_1 K_1\, dy + \int\limits_0^{l_2} u_2 K_2\, dy\right),\end{gathered} \tag{5.25}$$

where

$$E_3(t) = \frac{1}{2}\int\limits_{-l_1}^{0} K_1^2\, dy + \frac{1}{2}\int\limits_0^{l_2} K_2^2\, dy. \tag{5.26}$$

It can be easily deduced from these relations that $\int\limits_{-l_1}^{0} K_1^2\, dy$ and $\int\limits_0^{l_2} K_2^2\, dy$ are bounded for any finite t when $B_2 = 0$. It can be done with the help of elementary inequality $ab \leqslant \varepsilon a^2/2 + b^2/2\varepsilon$,

$\forall \varepsilon > 0$. Here it is difficult to obtain an inequality of type (2.10) or (4.10) from (5.25) and (5.26). The point is that the Friedrichs inequalities (2.9) does not hold for the functions $K_j(y,t)$. However, they are satisfied if the mean values $\int\limits_{-l_1}^{0} K_1(y,t)\,dy = 0$ and $\int\limits_{0}^{l_2} K_2(y,t)\,dy = 0$. It follows from a more general Poincare inequality

$$\int\limits_a^b f^2(x)\,dx \leqslant \frac{2}{b-a}\left(\int\limits_a^b f(x)\,dx\right)^2 + 2(b-a)^2\int\limits_a^b f'^2(x)\,dx.$$

However, the mean values are non-zero here. Therefore, this procedure does not allow us to determine the rate of convergence of $K_j(y,t)$ to zero when condition (2.12) is satisfied.

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