One approach to finding the resultant of two entire functions

A.M. Kytmanov, Y.M. Naprienko

In this paper, using the methods of complex analysis, introduced the concept of resultant for entire functions of certain classes and set its properties. This concept is compared with the classical Sylvester resultant.

Keywords: resultant, entire function of finite order of growth.

Introduction

Classical recurrent Newton's identities connect power sums of polynomial roots and its coefficients. The identities could be obtained with Cauchy's integral formula. This fact let us to expand class of functions for which recurrent identities is valid. Namely, for the class of entire functions of finite order of growth we can get relations between coefficients of function's Taylor expansion and power sums of zeros in negative power [1].

In the paper, using the methods of complex analysis, we introduce the concept of resultant for entire functions and set its properties. This concept is compared with the classical Sylvester resultant. Benefit of the generalization follows from the equations of chemical kinetics, where arise functions that are exponential polynomials [2].

Let $f_n(z)$ and $g_m(z)$ — polynomials of degrees n and m , respectively, with coefficients from the field of complex numbers $\mathbb{C}$:

$$f_n(z) = a_0 + a_1z + \dots + a_nz^n,$$

$$g_m(z) = b_0 + b_1z + \dots + b_mz^m.$$

Classical resultant for polynomials $f_n(z)$ и $g_m(z)$ usually defined as determinant of Sylvester matrix of order $n + m$ (for example, [3]):

$$D_{n,m} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_0 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_0 & \dots & b_m \end{bmatrix}. \quad (1)$$

The determinant (1) becomes zero if and only if either polynomials have a common root or leading coefficients a_n and b_m are zero simultaneously.

if $a_n \neq 0$, then

$$D_{n,m} = a_n^m \prod_{k=1}^n g_m(\alpha_k),$$

i.e

$$\prod_{k=1}^n g_m(\alpha_k) = \frac{1}{a_n^m} D_{n,m}, \quad (2)$$

where $\alpha_1, \dots, \alpha_n$ are roots of polynomial $f_n(z)$.

Denote

$$R(f_n, g_m) = \prod_{k=1}^n g_m(\alpha_k). \quad (3)$$

For polynomials $f_n(x)$ and $g_m(x)$ for $a_n = 1$ expression $R(f_n, g_m)$ is as follows:

$$R(f_n, g_m) = D_{n,m}.$$

Our goal is to gain analogous equality (3) for entire functions $f(z)$ и $g(z)$. The main difficulty lies in the fact that entire function may have infinity number of zeros, so product (3) may have no finite limit.

Using integral representation we obtain on the one hand power sums of zeros, and the other expression, depending on coefficients of Taylor series of functions $f(z)$ и $g(z)$.

Then, using classical Newton's identities, we obtain product of values one function in zeros of second function.

1 Предварительные сведения

1. First of all we recall Hadamard factorization theorem (e.g. [4, ch. 8]). Let $f(z)$ be such an entire function that there is exist a number A , that if $|z| = R \rightarrow \infty$ then follows:

$$f(z) = O(e^{R^A}),$$

then $f(z)$ is called a function of finite order of growth, and infimum of numbers A is called order of growth ρ of function $f(z)$.

Теорема 1 (Hadamard). *If $f(z)$ is a entire function of finite order of growth ρ with zeros $\alpha_1, \alpha_2, \dots$, then*

$$f(z) = z^s e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \dots + \frac{z^p}{p\alpha_n^p}}, \quad (4)$$

where $Q(z)$ is a polynomial of degree $q \leq \rho$, s is multiplicity of zero $f(z)$ at the point 0, and number $p \leq \rho$ (every zero considered as many times, as its multiplicity).

Expression

$$\Phi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \dots + \frac{z^p}{p\alpha_n^p}} \quad (5)$$

is called canonical product, and number p is the genus of canonical product.

2. There is statement about connection of coefficients of function and its logarithm [1, ch. 1]:

Лемма 1 (about connection of Taylor coefficients). *Let $f(z)$ be holomorphic in the neighborhood of zero, and $f(0) \neq 0$. Denote $g(z) = \ln f(z)$, f and g have next Taylor series:*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \dots + a_n z^n + \dots, \quad (6)$$

$$g(z) = \ln f(z) = \sum_{k=0}^{\infty} c_k z^k = c_0 + c_1 z + \dots + c_n z^n + \dots$$

Then next recurrent relations is valid:

$$k a_k = \sum_{j=0}^{k-1} (k-j) a_j c_{k-j}, \quad k \geq 1.$$

Moreover:

$$c_k = \frac{(-1)^{k-1}}{k a_0^k} \begin{bmatrix} a_1 & a_0 & 0 & \dots & 0 \\ 2a_2 & a_1 & a_0 & \dots & 0 \\ 3a_3 & a_2 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ k a_k & a_{k-1} & a_{k-2} & \dots & a_1 \end{bmatrix},$$

$$a_k = \frac{a_0}{k!} \begin{bmatrix} c_1 & -1 & 0 & \dots & 0 \\ 2c_2 & c_1 & -2 & \dots & 0 \\ 3c_3 & 2c_2 & c_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ k c_k & (k-1)c_{k-1} & (k-2)c_{k-2} & \dots & c_1 \end{bmatrix}.$$

We note, that in conditions of the lemma, the function $g(z)$ is also holomorphic in some neighborhood of zero, since $f(0) \neq 0$.

3. Let $s_0, s_1, s_2 \dots$ be a sequence of complex numbers. The sequence define infinite Hankel matrix:

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & s_4 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (7)$$

Successive minors of the matrix S we denote as $D_1, D_2, D_3, \dots$:

$$D_p = |s_{j+k}|_0^{p-1}, \quad p = 1, 2, \dots$$

If for each $p \in \mathbb{N}$ exist a non-zero minor of matrix S of order p , then the matrix has an infinite rank. If, starting with some p , all minors of higher orders are equal to zero, then the matrix has a finite rank. The smallest such number is called the rank of a matrix. Here are two statements about matrix S of finite order p (e.g [5, ch. 16, §10])

Следствие 1. *If an infinite Hankel matrix S of form (7) has a finite rank p , then $D_p \neq 0$.*

Теорема 2. *A matrix S has a finite rank p if, and only if a sum of series*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots$$

is a rational function of z . In this case, rank of matrix S coincides with the number of poles of function $R(z)$, counting each root as many times as its multiplicity.

2 Вспомогательные утверждения

Let $g(z)$ is an entire function, and $f(z)$ is an entire function of finite order of growth.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

Consider a circle $C_R = \{z \in \mathbb{C} : |z| < R\}$ of sufficiently large radius to contain a finite zeros (with multiplicities) of function $f(z)$: $\{\alpha_1, \dots, \alpha_l\}$. On the edge γ_R of the circle there are no zeros. And consider a circle C_r , where $R = \frac{1}{r}$, with edge γ_r .

Лемма 2. *We have the following equations:*

$$\int_{\gamma_r} \frac{g(\frac{1}{z})df(\frac{1}{z})}{f(\frac{1}{z})} = - \int_{\gamma_R} \frac{g(w)df(w)}{f(w)} = -(2\pi i) \sum_{n=1}^l g(\alpha_n). \quad (8)$$

Доказательство. First equality follows from the change of orientation when we substitute $w = \frac{1}{z}$.

The second in (8) is a consequence of theorem of logarithmic residue. □

Лемма 3. *Let $g(z)$ be an entire function, and $f(z)$ is a polynomial of degree n ,*

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = b_0 + b_1 z + \dots + b_k z^k + \dots,$$

$$f(z) = \sum_{k=0}^n a_k z^k = a_0 + a_1 z + \dots + a_n z^n, \quad a_n \neq 0.$$

Then we have the equation:

$$\int_{\gamma_r} \frac{g(\frac{1}{z})df(\frac{1}{z})}{f(\frac{1}{z})} = -(2\pi i)nb_0 + (2\pi i) \sum_{k=1}^{\infty} k\tilde{a}_k b_k,$$

where $\tilde{a}_1, \tilde{a}_2, \dots$ — coefficients of Taylor series of function $\ln(a_n + a_{n-1}z + \dots + a_0z^n)$, and γ_r is the edge of circle C_r of sufficiently small radius r so as not to contain zeros of function $f(1/z)$, and where $\ln(a_n + a_{n-1}z + \dots + a_0z^n)$ is holomorphic. Moreover, series in the equation converges absolutely.

Доказательство. We have

$$\begin{aligned} \int_{\gamma_r} \frac{g(\frac{1}{z})df(\frac{1}{z})}{f(\frac{1}{z})} &= \int_{\gamma_r} \frac{g(\frac{1}{z})d(\frac{1}{z^n}(a_n + a_{n-1}z + \dots + a_0z^n))}{\frac{1}{z^n}(a_n + a_{n-1}z + \dots + a_0z^n)} = \\ &= \int_{\gamma_r} \frac{g(\frac{1}{z}) \left(d(\frac{1}{z^n})(a_n + \dots + a_0z^n) + \frac{1}{z^n}(a_n + \dots + a_0z^n)'dz \right)}{\frac{1}{z^n}(a_n + \dots + a_0z^n)} = \\ &= \int_{\gamma_r} \frac{g(\frac{1}{z}) \left(-\frac{n}{z^{n+1}}(a_n + \dots + a_0z^n)dz + \frac{1}{z^n}(a_n + \dots + a_0z^n)'dz \right)}{\frac{1}{z^n}(a_n + \dots + a_0z^n)} = \\ &= -n \int_{\gamma_r} \frac{g(\frac{1}{z})dz}{z} + \int_{\gamma_r} \frac{g(\frac{1}{z})(a_n + \dots + a_0z^n)'dz}{(a_n + \dots + a_0z^n)} = -(2\pi i)nb_0 + \int_{\gamma_r} g\left(\frac{1}{z}\right) (\ln(a_n + \dots + a_0z^n))'dz = \\ &= -(2\pi i)nb_0 + \sum_{j=0}^{\infty} b_j \sum_{k=1}^{\infty} k\tilde{a}_k \int_{\gamma_r} z^{k-j-1}dz = -(2\pi i)nb_0 + (2\pi i) \sum_{k=1}^{\infty} k\tilde{a}_k b_k. \end{aligned}$$

The last series converges absolutely by the root test:

$$\lim_{k \rightarrow \infty} \sqrt[k]{k|\tilde{a}_k b_k|} = 0,$$

since $\sqrt[k]{k|\tilde{a}_k|}$ are bounded, and $\sqrt[k]{|b_k|}$ tend to zero. □

Here is a formula for raising power series to a power.

Let A and B be power series with non-zero radius of converges.

$$A = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots, \quad a_0 \neq 0,$$

$$B = A^n = \sum_{k=0}^{\infty} b_k z^k = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n + \dots$$

where n is natural number.

Лемма 4. *The following equation holds*

$$b_k = \frac{a_0^{n-k}}{k!} \begin{vmatrix} na_1 & -1a_0 & 0 & \dots & 0 \\ 2na_2 & (n-1)a_1 & -2a_0 & \dots & 0 \\ 3na_3 & (2n-1)a_2 & (n-2)a_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kna_k & ((k-1)n-1)a_{k-1} & ((k-2)n-2)a_{k-2} & \dots & (n-(k-1))a_1 \end{vmatrix}, \quad (9)$$

$k \geq 1$.

Доказательство. Differentiate $A^n = B$:

$$nA'A^{n-1} = B'$$

and multiply both sides by A . Then

$$nA'B = AB',$$

$$\begin{aligned} n \sum_{k=1}^{\infty} ka_k z^{k-1} \sum_{k=0}^{\infty} b_k z^k &= \sum_{k=0}^{\infty} a_k z^k \sum_{k=1}^{\infty} kb_k z^{k-1}, \\ \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} n(i+1)a_{i+1}b_{k-i-1} \right) z^k &= \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} (k-i)a_i b_{k-i} \right) z^k. \end{aligned}$$

Equal series have equal coefficients of each degree:

$$\sum_{i=0}^{k-1} n(i+1)a_{i+1}b_{k-i-1} = \sum_{i=1}^{k-1} (k-i-1)a_{i+1}b_{k-i-1} + ka_0b_k.$$

From this, we find a recurrent relations

$$ka_0b_k = \sum_{i=0}^{k-1} (n(i+n) - (k-i-1)) a_{i+1}b_{k-i-1} = \sum_{i=1}^k (ni - (k-i)) a_i b_{k-i}.$$

Using the recurrent relations we obtain system of equations:

$$\begin{pmatrix} -1a_0 & 0 & \dots & 0 & 0 \\ (n-1)a_1 & -2a_0 & \dots & 0 & 0 \\ (2n-1)a_2 & (n-2)a_1 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \vdots & \vdots \\ ((k-1)n-1)a_{k-1} & ((k-2)n-2)a_{k-2} & \dots & (n-(k-1))a_1 & -ka_0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{pmatrix} =$$

$$= \begin{pmatrix} -na_1b_0 \\ -2na_2b_0 \\ -3na_3b_0 \\ \vdots \\ -kna_kb_0 \end{pmatrix}.$$

Determinant of matrix is equal to $\Delta = (-1)^k a_0^k k!$. Using Cramer's rule, after some manipulations we get

$$b_k = \frac{a_0^{n-k}}{k!} \begin{vmatrix} na_1 & -1a_0 & 0 & \dots & 0 \\ 2na_2 & (n-1)a_1 & -2a_0 & \dots & 0 \\ 3na_3 & (2n-1)a_2 & (n-2)a_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kna_k & ((k-1)n-1)a_{k-1} & ((k-2)n-2)a_{k-2} & \dots & (n-(k-1))a_1 \end{vmatrix}.$$

□

Comment 1. An important case is when $a_0 = 0$, $a_1 \neq 0$. Then

$$b_0 = b_1 = \dots = b_{n-1} = 0$$

$$b_{n+k} = \frac{a_1^{n-k}}{k!} \begin{bmatrix} na_2 & -1a_1 & 0 & \dots & 0 \\ 2na_3 & (n-1)a_2 & -2a_1 & \dots & 0 \\ 3na_4 & (2n-1)a_3 & (n-2)a_2 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kna_{k+1} & ((k-1)n-1)a_k & ((k-2)n-2)a_{k-1} & \dots & (n-(k-1))a_2 \end{bmatrix}.$$

There formulas is obtained similarly (9).

Comment 2. Lemma 4 is valid for formal power series.

Comment 3. Lemma 4 is valid when degree n is a complex number α . If $(a_0 \neq 0)$, then we can assume

$$A^\alpha = e^{\alpha \ln A}.$$

A function $\ln A$ is holomorphic in neighborhood of zero.

3 Основные утверждения

Теорема 3. Let $g(z)$ be an entire function, and $f(z)$ is polynomial of degree n with roots $\alpha_1, \dots, \alpha_n$.

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = b_0 + b_1 z + \dots + b_k z^k + \dots,$$

$$f(z) = \sum_{k=0}^n a_k z^k = a_0 + a_1 z + \dots + a_n z^n, \quad a_n \neq 0.$$

Then next equation holds

$$\sum_{i=1}^n g(\alpha_i) = \lim_{k \rightarrow \infty} \frac{1}{a_n^k} \begin{bmatrix} nb_0 & b_1 & b_2 & \dots & b_k \\ a_{n-1} & a_n & 0 & \dots & 0 \\ 2a_{n-2} & a_{n-1} & a_n & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ ka_{n-k} & a_{n-k+1} & a_{n-k+2} & \dots & a_n \end{bmatrix},$$

and the limit exists and is finite.

Доказательство. If we take sufficiently large area, which contains all the roots of polynomial $f(z)$, the from lemmas 2, 3 follows

$$\sum_{k=1}^n g(\alpha_k) = nb_0 - \sum_{k=1}^{\infty} k\tilde{a}_k b_k,$$

where $\tilde{a}_1, \tilde{a}_2, \dots$ are coefficients of Taylor series for function $\ln(a_n + a_{n-1}z + \dots + a_0z^n)$.

From lemma 1 about connection of coefficients of function and its logarithm, we have

$$\tilde{a}_k = \frac{(-1)^{k-1}}{ka_n^k} \begin{bmatrix} a_{n-1} & a_n & 0 & \dots & 0 \\ 2a_{n-2} & a_{n-1} & a_n & \dots & 0 \\ 3a_{n-3} & a_{n-2} & a_{n-1} & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ ka_{n-k} & a_{n-k+1} & a_{n-k+2} & \dots & a_{n-1} \end{bmatrix}.$$

Using this representation, we get

$$\sum_{k=1}^n g(\alpha_k) = nb_0 + \sum_{k=1}^{\infty} \frac{(-1)^k b_k}{a_n^k} \begin{bmatrix} a_{n-1} & a_n & 0 & \dots & 0 \\ 2a_{n-2} & a_{n-1} & a_n & \dots & 0 \\ 3a_{n-3} & a_{n-2} & a_{n-1} & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ ka_{n-k} & a_{n-k+1} & a_{n-k+2} & \dots & a_{n-1} \end{bmatrix}.$$

We can write the partial sum of the last series as follows:

$$\frac{1}{a_n^k} \begin{bmatrix} 0 & b_1 & b_2 & \dots & b_m \\ a_{n-1} & a_n & 0 & \dots & 0 \\ 2a_{n-2} & a_{n-1} & a_n & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ ka_{n-k} & a_{n-k+1} & a_{n-k+2} & \dots & a_n \end{bmatrix}.$$

The proof of equality is obtained by expansion of first row and using Laplace expansion. If we add nb_0 instead a zero in the top left corner, after passing to the limit we have

$$\sum_{k=1}^n g(\alpha_k) = \lim_{k \rightarrow \infty} \frac{1}{a_n^k} \begin{bmatrix} nb_0 & b_1 & b_2 & \dots & b_k \\ a_{n-1} & a_n & 0 & \dots & 0 \\ 2a_{n-2} & a_{n-1} & a_n & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ ka_{n-k} & a_{n-k+1} & a_{n-k+2} & \dots & a_n \end{bmatrix}.$$

□

Теорема 4. Let $g(z)$ be an entire function, and $f(z)$ is polynomial of degree n with roots $\alpha_1, \dots, \alpha_n$. Then the following equation is true

$$\prod_{k=1}^n g(\alpha_k) = \frac{1}{n!} \begin{bmatrix} S_1 & 1 & 0 & \dots & 0 \\ S_2 & S_1 & 2 & \dots & 0 \\ S_3 & S_2 & S_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{bmatrix},$$

where $S_k = \sum_{i=1}^n g^k(\alpha_i)$.

We can find $S_k = \sum_{i=1}^n g^k(\alpha_i)$ using theorem 3 and lemma 4.

Доказательство. We construct a polynomial of degree n with roots $g(\alpha_i)$, $1 \leq i \leq n$,

$$R(z) = \prod_{k=1}^n (z - g(\alpha_k)) = d_n + d_{n-1}z + \dots + z^n.$$

It's obvious that constant term of $R(z)$ is $(-1)^n \prod_{k=1}^n g(\alpha_k)$. On the other hand known classical Newton's formula for the coefficients of the polynomial expression through the power sums of its roots (e.g [3]):

$$d_n = \frac{(-1)^n}{n!} \begin{bmatrix} S_1 & 1 & 0 & \dots & 0 \\ S_2 & S_1 & 2 & \dots & 0 \\ S_3 & S_2 & S_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{bmatrix},$$

where $S_k = \sum_{i=1}^n g^k(\alpha_i)$ is a power sum of polynomial roots.

Since free term is equal to the resultant and the determinant, we have proof of theorem.

4. □

Example 1. We calculate a resultant for function $g(z) = e^z - 1$ and polynomial $f(z) = z^2 - 3z$.

$$S_1 = \lim_{k \rightarrow \infty} \begin{bmatrix} 0 & 1 & \frac{1}{2!} & \dots & \frac{1}{k!} \\ -3 & 1 & 0 & \dots & 0 \\ 0 & -3 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

By expansion first column many times, we have

$$S_1 = \lim_{k \rightarrow \infty} 3 \left(1 + \frac{3}{2!} + \frac{3^2}{3!} + \cdots + \frac{3^{k-1}}{k!} \right) = e^3 - 1,$$

$$g^2(z) = (e^z - 1)^2 = e^{2z} - 2e^z + 1 = \sum_{n=1}^{\infty} \left(\frac{2^n - 2}{n!} \right) z^n.$$

Calculating in the same manner S_2 , we get

$$S_2 = e^6 - 2e^3 + 1.$$

Then the resultant is equal to

$$g(0) \cdot g(3) = \frac{1}{2!} \begin{bmatrix} e^3 - 1 & 1 \\ e^6 - 2e^3 + 1 & e^3 - 1 \end{bmatrix} = 0.$$

Example 2. Let

$$g(z) = \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$f(z) = \left(z - \frac{\pi}{2} \right) (z - \pi) = \frac{\pi^2}{2} - \frac{3\pi}{2}z + z^2$$

In this case $\alpha_1 = \pi$, $\alpha_2 = \frac{\pi}{2}$. Let's obtain the resultant by using power sums (theorem 3):

$$S_1 = \sum_{i=1}^2 \cos(\alpha_i) = \lim_{k \rightarrow \infty} \begin{bmatrix} 2 & 0 & -\frac{1}{2!} & \cdots & 0 & \frac{(-1)^k}{(2k)!} \\ -\frac{3\pi}{2} & 1 & 0 & \cdots & 0 & 0 \\ \pi^2 & -\frac{3\pi}{2} & 1 & \cdots & 0 & 0 \\ 0 & \frac{\pi^2}{2} & -\frac{3\pi}{2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{3\pi}{2} & 1 \end{bmatrix}$$

By expansion first row, we have

$$\lim_{k \rightarrow \infty} \left(2 + \sum_{n=1}^k \frac{(-1)^n}{(2n)!} \begin{bmatrix} -\frac{3\pi}{2} & 1 & 0 & \cdots & 0 \\ \pi^2 & -\frac{3\pi}{2} & 1 & \cdots & 0 \\ 0 & \frac{\pi^2}{2} & -\frac{3\pi}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{3\pi}{2} \end{bmatrix} \right),$$

where determinant of matrix has order n . Denote a determinant as D_n . We prove by induction, that $D_n = (-1)^n \left(\frac{\pi^n}{2^n} + \pi^n \right)$. We have $D_1 = -\frac{3\pi}{2}$, it's base of induction.

We assume that for $m \leq n$ proven formula. For $n + 1$ enough to expand on the first row and get

$$D_{n+1} = -\frac{3\pi}{2}D_n - \frac{\pi^2}{2}D_{n-1} = (-1)^{n+1}\frac{3\pi}{2}\left(\frac{\pi^n}{2^n} + \pi^n\right) + (-1)^n\frac{\pi^2}{2}\left(\frac{\pi^{n-1}}{2^{n-1}} + \pi^{n-1}\right),$$

$$D_{n+1} = (-1)^{n+1}\left(\frac{\pi^{n+1}}{2^{n+1}} + \pi^{n+1}\right).$$

Considering this, we have

$$\lim_{k \rightarrow \infty} \left(2 + \sum_{n=1}^k \frac{(-1)^n}{(2n)!} D_{2n}\right) = \lim_{k \rightarrow \infty} \left(\sum_{n=0}^k \frac{(-1)^n}{(2n)!} \pi^{2n} + \sum_{n=0}^k \frac{(-1)^n}{(2n)!} \frac{\pi^{2n}}{2^n}\right) = \cos\left(\frac{\pi}{2}\right) + \cos(\pi) = -1.$$

Analogically,

$$S_2 = \cos^2\left(\frac{\pi}{2}\right) + \cos^2(\pi) = 1.$$

Then the resultant is equal to determinant (4)

$$\prod_{i=1}^2 \cos(\alpha_i) = \frac{1}{2!} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0.$$

4 СВЯЗЬ С КЛАССИЧЕСКИМ РЕЗУЛЬТАНТОМ

We have proved that limit in theorem 4 exists, and it allows us to consider resultant of function in roots of polynomial, which is equivalent to limit in the classical resultant of the variable m , if polynomial $g_m(z)$ is Taylor polynomial (of degree m) for an entire function $g(z)$.

Следствие 2. *The following formula holds*

$$\prod_{k=1}^n g(\alpha_k) = \lim_{m \rightarrow \infty} \frac{1}{a_n^m} \begin{bmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_0 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_0 & \dots & b_m \end{bmatrix}. \quad (10)$$

Доказательство. From (1), (3) we have

$$\lim_{m \rightarrow \infty} \prod_{k=1}^n g_m(\alpha_k) = \lim_{m \rightarrow \infty} \frac{1}{a_n^m} \begin{bmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_0 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_0 & \dots & b_m \end{bmatrix}.$$

That is

$$\prod_{k=1}^n g(\alpha_k) = \lim_{m \rightarrow \infty} \frac{1}{a_n^m} \begin{bmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_0 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_0 & \dots & b_m \end{bmatrix}.$$

□

Comment 4. Formula (10) could be obtained directly from the definition of resultant and formula (2), because $g_m \rightarrow g$ evenly in any circle. So $g_m(\alpha_k) \rightarrow g(\alpha_k)$ as $m \rightarrow \infty$ for any number α_k , $k = 1, \dots, n$.

Consider concerning examples 1,2. In example 1, it's obvious, that formula (10) gives 0. In example 2, resultant is equal to limit of Sylvester determinants

$$\prod_{k=1}^n g(\alpha_k) = \lim_{m \rightarrow \infty} \begin{bmatrix} 1 & 0 & \frac{1}{2!} & 0 & \dots & \frac{(-1)^k}{(2k)!} \\ 0 & 1 & 0 & \frac{1}{2!} & \dots & 0 \\ \frac{\pi^2}{2} & -\frac{3\pi}{2} & 1 & 0 & \dots & 0 \\ 0 & \frac{\pi^2}{2} & -\frac{3\pi}{2} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Its calculation is much more complicated task then using theorem 4.

5 Factorization

We can extend class of functions by factorizing function and applying theorem ?? for function and polynomial.

Consider an entire function $f(z)$ of a finite order of growth ρ with finite number of zeros $\alpha_1, \dots, \alpha_m$ satisfying Hadamard theorem. Then

$$f(z) = z^s e^{Q(z)} \prod_{n=1}^m \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \dots + \frac{z^q}{q\alpha_n^q}},$$

where $Q(z)$ is polynomial with degree $p = \rho$. We assume $s = 0$. Otherwise, it's easy to pick out this factor.

Since canonical product for f has order 0, we can write function as follows:

$$f(z) = e^{-Q(z)}P(z), \quad (11)$$

where $Q(z)$ is polynomial of degree $p = \rho$, and $P(z)$ is polynomial of degree m .

Without loss of generality we can assume that free term of function $f(z)$ is equal to one. Then polynomial $Q(z)$ has a zero free term, and $P(z)$ has a free term is equal to one. We have

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = 1 + a_1 z + \cdots + a_n z^n + \cdots,$$

$$Q(z) = \sum_{k=1}^p d_k z^k = d_1 z + \cdots + d_p z^p,$$

$$P(z) = \sum_{k=0}^m b_k z^k = 1 + b_1 z + \cdots + b_m z^m.$$

Take the logarithm of both sides (11) in a neighborhood of zero and get

$$\ln f(z) + Q(z) = \ln P(z). \quad (12)$$

Let

$$\ln f(z) = \sum_{k=1}^{\infty} \tilde{a}_k z^k = \tilde{a}_1 z + \cdots + \tilde{a}_n z^n + \cdots,$$

$$\ln P(z) = \sum_{k=1}^{\infty} \tilde{b}_k z^k = \tilde{b}_1 z + \cdots + \tilde{b}_n z^n + \cdots.$$

Consider derivative $S(z) = (\ln f(\frac{1}{z}))'$. Since

$$\ln f\left(\frac{1}{z}\right) = \sum_{k=1}^{\infty} \tilde{a}_k z^{-k} = \tilde{a}_1 \frac{1}{z} + \cdots + \tilde{a}_n \frac{1}{z^n} + \cdots,$$

then

$$S(z) = - \sum_{k=1}^{\infty} k \tilde{a}_k z^{-k-1}.$$

On the other hand, using equation (12), we have

$$S(z) = \left(\ln f\left(\frac{1}{z}\right) \right)' = \left(\ln P\left(\frac{1}{z}\right) \right)' + \left(Q\left(\frac{1}{z}\right) \right)'.$$

Obviously, function $\left(Q\left(\frac{1}{z}\right)\right)'$ has a pole at point $z = 0$ of order $p + 1$. Polynomial

$P(z) = b_m \prod_{k=1}^m (z - \alpha_k)$, so

$$P\left(\frac{1}{z}\right) = \frac{b_m}{z^m} \prod_{k=1}^m (1 - z\alpha_k).$$

$$\left(\ln P\left(\frac{1}{z}\right)\right)' = -\sum_{k=1}^m \frac{\alpha_k}{1 - z\alpha_k} - \frac{m}{z}.$$

Hence, $S(z)$ is rational function and number of its poles (with multiplicity) equal to $m + p + 1$. Applying theorem 2 to function $S(z)$, we obtain that rank of infinite Hankel matrix $\|(k + s)\tilde{a}_{k+s}\|_1^\infty$, build from the coefficients $k\tilde{a}_k$ of function $S(z)$ is equal to $m + p + 1$. Moreover, the main minor of order $m + p - 1$

$$D_{m+p+1} = |(k + s)\tilde{a}_{k+s}|_1^{m+p+1} \neq 0,$$

since 1. If we remove from $S(z)$ terms that form poles at point 0, then we get rational function $\varphi(z)$, which has exactly m poles

$$\varphi(z) = \sum_{k=1}^m \frac{\alpha_k}{z\alpha_k - 1}.$$

Functions $\varphi(z)$ and $-S(z)$ have identical Laurent coefficients, starting from $(p + 1)$ term. Therefore determinant

$$\Delta_m = |(s + k)\tilde{a}_{s+k}|_{p+1}^{m+p} \neq 0. \quad (13)$$

Equation coefficients of equal powers in (12), we find

$$\begin{cases} \tilde{a}_1 + d_1 = \tilde{b}_1, \\ \tilde{a}_2 + d_2 = \tilde{b}_2, \\ \dots \\ \tilde{a}_p + d_p = \tilde{b}_p, \\ \tilde{a}_{p+1} = \tilde{b}_{p+1}, \\ \tilde{a}_{p+2} = \tilde{b}_{p+2}, \\ \dots \\ \tilde{a}_m = \tilde{b}_m, \\ \dots \end{cases}$$

Using lemma 1 about connection of coefficients of function and its logarithm, we have:

$$\begin{cases} b_1 = b_0 \tilde{b}_1, \\ b_2 = 2b_0 \tilde{b}_2 + b_1 \tilde{b}_1, \\ b_3 = 3b_0 \tilde{b}_3 + 2b_1 \tilde{b}_2 + b_2 \tilde{b}_1, \\ \dots \\ b_m = mb_0 \tilde{b}_m + (m-1)b_1 \tilde{b}_{m-1} + \dots + b_{m-1} \tilde{b}_1, \\ 0 = (m+1)b_0 \tilde{b}_{m+1} + \dots + b_m \tilde{b}_1, \\ 0 = (m+2)b_0 \tilde{b}_{m+2} + \dots + b_m \tilde{b}_2, \\ \dots \\ 0 = (m+p+1)b_0 \tilde{b}_{m+p+1} + \dots + b_m \tilde{b}_{p+1}, \\ 0 = (m+p+2)b_0 \tilde{b}_{m+p+2} + \dots + b_m \tilde{b}_{p+2}, \\ \dots \\ 0 = (2m+p)b_0 \tilde{b}_{2m+p} + \dots + b_m \tilde{b}_{m+p}. \\ \dots \end{cases}$$

Take m equations, starting with $(p+1)$ -th. In respect that $b_0 = 1$, and every $\tilde{b}_i = \tilde{a}_i$ ($i > p$), we obtain the following system of equations:

$$\begin{pmatrix} (m+p)\tilde{a}_{m+p} & \dots & (p+1)\tilde{a}_{p+1} \\ (m+p+1)\tilde{a}_{m+p+1} & \dots & (p+2)\tilde{a}_{p+2} \\ \dots & & \\ (2m+p-1)\tilde{a}_{2m+p-1} & \dots & (m+p)\tilde{a}_{m+p} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} = \begin{pmatrix} -(m+p+1)\tilde{a}_{m+p+1} \\ -(m+p+2)\tilde{a}_{m+p+2} \\ \dots \\ -(2m+p)\tilde{a}_{2m+p} \end{pmatrix}$$

By Cramer's rule, we find

$$b_k = \frac{- \begin{bmatrix} (m+p)\tilde{a}_{m+p} & \dots & (m+p+1)\tilde{a}_{m+p+1} & \dots & (p+1)\tilde{a}_{p+1} \\ (m+p+1)\tilde{a}_{m+p+1} & \dots & (m+p+2)\tilde{a}_{m+p+2} & \dots & (p+2)\tilde{a}_{p+2} \\ \dots & & & & \\ (2m+p-1)\tilde{a}_{2m+p-1} & \dots & (2m+p)\tilde{a}_{2m+p} & \dots & (m+p)\tilde{a}_{m+p} \end{bmatrix}}{\begin{bmatrix} (m+p)\tilde{a}_{m+p} & \dots & (p+1)\tilde{a}_{p+1} \\ (m+p+1)\tilde{a}_{m+p+1} & \dots & (p+2)\tilde{a}_{p+2} \\ \dots & & \\ (2m+p-1)\tilde{a}_{2m+p-1} & \dots & (m+p)\tilde{a}_{m+p} \end{bmatrix}}$$

In numerator k -th column is replaced by column of free terms. Determinant in denominator is not equal to zero by property (13). $\square$

Example 3. Factorize function

$$f(z) = e^{2z}(1-z)(1+z).$$

According to the above formulas, we have first $2m+p$ coefficients of function's logarithm, where m is number of zeros, and p is order of growth. Take the logarithm of both sides

and expanding in series around zero, we have

$$\ln f(z) = 2z - z^2 - \frac{z^4}{2} - \frac{z^6}{3} + \dots$$

From here we find

$$b_1 = -\frac{\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}}{\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}} = 0,$$
$$b_2 = -\frac{\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}}{\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}} = -1.$$

So we can write polynomial with function's roots: $F(z) = 1 - z^2$. And we can use the resultant to find common zeros.

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