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Microlocal Study of Lefschetz Fixed Point Formulas

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The aim of this short paper is to introduce our recent study on Lefschetz fixed point formulas over singular varieties. In particular, we generalize Kashiwara's theory of characteristic cycles by introducing new Lagrangian cycles associated with endomorphisms of constructible sheaves. Some examples related with Schubert varieties and toric hypersurfaces will also be given.

Key words: *singular varieties, Lefschetz fixed point formulas, Lagrangian cycles, Schubert varieties.*

Introduction

In this paper, we introduce our recent study of Lefschetz fixed point formulas in [15]. First of all, let us consider the (classical) Lefschetz fixed point formulas for morphisms $\phi\colon X \longrightarrow X$ of real analytic manifolds X. We denote the fixed point set of ϕ by $M = \{x \in X \mid \phi(x) = x\} \subset X$ (since here we mainly consider the case where the fixed point set is smooth, we used the symbol M) and let $M = \bigsqcup_{i\in I} M_i$ be the decomposition of M into connected components. Then it is well-known that if X and M are compact the global Lefschetz number of ϕ

$$\mathrm{tr}(\phi) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(X;\mathbb{C}_X) \xrightarrow{\phi^*} H^j(X;\mathbb{C}_X)\} \in \mathbb{C}$$

is a sum of some numbers $c(\phi)_{M_i} \in \mathbb{C}$ associated with the fixed point components M_i's. The number $c(\phi)_{M_i}$ is called the local contribution from M_i. But if the fixed point component M_i is "higher-dimensional", it is a very hard task to compute the local contribution in general. In our paper [15], we studied more general problems for the Lefschetz numbers of hypercohomology groups of constructible sheaves and overcame this difficulty for smooth fixed point components M_i's. In particular, we obtained some Lefschetz fixed point formulas over singular varieties (see e.g. Corollary 1). Moreover new Lagrangian cycles in the cotangent bundles T^*M_i, which we call Lefschetz cycles, were introduced and their functorial properties were studied precisely. This result generalizes Kashiwara's theory of characteristic cycles. For details, see [15].

1. Preliminary Notions and Results

In this note, we essentially follow the terminology in [11]. For example, for a topological space X, we denote by $\mathbf{D}^b(X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. Since

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we focus our attention on Lefschetz fixed point formulas for constructible sheaves in this paper, we treat here only real analytic manifolds and morphisms. Now let X be a real analytic manifold. We denote by $\mathbf{D}^b_{\mathbb{R}-c}(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of the bounded complexes of sheaves whose cohomology sheaves are $\mathbb{R}$-constructible (see [11, Chapter VIII] for the definition). Denote also by $\omega_X \simeq or_X[\dim X] \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ the dualizing complex of X. Let $\phi\colon X \longrightarrow X$ be an endomorphism of the real analytic manifold X. Then our initial data is a pair (F, Φ) of $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ and a morphism $\Phi\colon \phi^{-1}F \longrightarrow F$ in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. If the support $\mathrm{supp}(F)$ of F is compact, $H^j(X;F)$ is a finite-dimensional vector space over $\mathbb{C}$ for any $j \in \mathbb{Z}$ and we can define the following number from (F,Φ).

Definition 1. *We set*

$$\mathrm{tr}(F,\Phi) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(X;F) \xrightarrow{\Phi} H^j(X;F)\} \in \mathbb{C},$$

where the morphisms $H^j(X;F) \xrightarrow{\Phi} H^j(X;F)$ are induced by

$$F \longrightarrow R\phi_*\phi^{-1}F \xrightarrow{\Phi} R\phi_*F..$$

We call $\mathrm{tr}(F,\Phi)$ the global trace (Lefschetz number) of the pair (F,Φ).

Now consider the fixed point set of $\phi\colon X \to X$ in X:

$$M := \{x \in X \mid \phi(x) = x\} \subset X.$$

Since we mainly consider the case where the fixed point set is smooth, we use the symbol M to express it. Note that if a compact group G is acting on X and ϕ is the left action of an element of G, then the fixed point set is smooth by Palais's theorem [17] (see also [6] for a survey on this subject). Also in this very general setting, Kashiwara [10] proved the following result.

Theorem 1 (Kashiwara [10])**.** *If $\mathrm{supp}(F)$ is compact, then there exists a class*

$$C(F,\Phi) \in H^0_{\mathrm{supp}(F)\cap M}(X;\omega_X) \simeq H^{\dim X}_{\mathrm{supp}(F)\cap M}(X; or_X)$$

such that the equality

$$\mathrm{tr}(F,\Phi) = \int_X C(F,\Phi) \tag{1}$$

holds. Here

$$\int_X \colon H_c^{\dim X}(X; or_X) \longrightarrow \mathbb{C}$$

is the morphism induced by the integral of differential $(\dim X)$-forms with compact support.

Let $M = \bigsqcup_{i\in I} M_i$ be the decomposition of M into connected components and $H^0_{\mathrm{supp}(F)\cap M}(X;\omega_X) = \bigoplus_{i\in I} H^0_{\mathrm{supp}(F)\cap M_i}(X;\omega_X)$, $C(F,\Phi) = \bigoplus_{i\in I} C(F,\Phi)_{M_i}$ the associated direct sum decomposition.

Definition 2. *When $\mathrm{supp}(F)\cap M_i$ is compact, we define a complex number $c(F,\Phi)_{M_i}$ by*

$$c(F,\Phi)_{M_i} := \int_X C(F,\Phi)_{M_i}$$

and call it the local contribution of the pair (F,Φ) from M_i.

With this notation, (1) is rewritten as

$$\mathrm{tr}(F,\Phi) = \sum_{i\in I} c(F,\Phi)_{M_i}.$$

Therefore the next important problem in the Lefschetz fixed point formula for constructible sheaves is to describe these local contributions $c(F,\Phi)_{M_i}$. However, the direct computation of local contributions is in general a very difficult task. Instead of directly considering the local contribution, let us first consider the following number $\mathrm{tr}(F|_{M_i}, \Phi|_{M_i})$, which is much more easily computed. Let M_i be a compact fixed point component such that $\mathrm{supp}(F) \cap M_i$ is compact.

Definition 3. *Consider the morphism (the restriction of $\Phi\colon \phi^{-1}F \longrightarrow F$ to M_i):*

$$\Phi|_{M_i}\colon F|_{M_i} \simeq (\phi^{-1}F)|_{M_i} \xrightarrow{\Phi} F|_{M_i}$$

and set

$$\mathrm{tr}(F|_{M_i}, \Phi|_{M_i}) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(M_i; F|_{M_i}) \xrightarrow{\Phi|_{M_i}} H^j(M_i; F|_{M_i})\}.$$

Then we can easily compute $\mathrm{tr}(F|_{M_i}, \Phi|_{M_i}) \in \mathbb{C}$ as follows. Let $M_i = \bigsqcup_{\alpha\in A} M_{i,\alpha}$ be a stratification of M_i by connected subanalytic manifolds $M_{i,\alpha}$ such that $H^j(F)|_{M_{i,\alpha}}$ is a locally constant sheaf for any $\alpha \in A$ and $j \in \mathbb{Z}$. Namely, we assume that the stratification $M_i = \bigsqcup_{\alpha\in A} M_{i,\alpha}$ is adapted to $F|_{M_i}$.

Definition 4. *For each $\alpha \in A$, we set*

$$c_\alpha := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(F)_{x_\alpha} \xrightarrow{\Phi|_{\{x_\alpha\}}} H^j(F)_{x_\alpha}\} \in \mathbb{C},$$

where x_α is a reference point of $M_{i,\alpha}$.

Then we have the following very useful result due to Goresky-MacPherson.

Proposition 1 (Goresky-MacPherson [5])**.** *We have*

$$\mathrm{tr}(F|_{M_i}, \Phi|_{M_i}) = \sum_{\alpha\in A} c_\alpha \cdot \chi_c(M_{i,\alpha}),$$

where χ_c is the Euler-Poincaré index with compact supports.

Therefore our main problem is:

Problem 1. *When does the following equality hold?*

$$c(F,\Phi)_{M_i} = \mathrm{tr}(F|_{M_i}, \Phi|_{M_i}). \tag{2}$$

For this problem, we have the following known results.

Example 1. (i) (Kashiwara-Schapira [11]) *If X, ϕ and F are all complex analytic, M_i is a point $\{\mathrm{pt}\}$ such that 1 is not an eigenvalue of the linear map $\phi'\colon T_{M_i}X \simeq \mathbb{C}^{\dim X} \longrightarrow T_{M_i}X \simeq \mathbb{C}^{\dim X}$, then* (2) *holds.*

(ii) (Goresky-MacPherson [5]) *If $\phi\colon X \longrightarrow X$ is weakly hyperbolic (see [5] for the definition) along M_i, then* (2) *holds. In particular, if ϕ is of finite order, then* (2) *holds.*

2. Localization Theorems and Their Applications

In this section, we give an answer to Problem 1 by partially generalizing the results of Kashiwara-Schapira [11] and Goresky-MacPherson [5] in Example 1. Since we always consider the same fixed point component M_i in this section, we denote M_i, $c(F,\Phi)_{M_i}$ etc. simply by M, $c(F,\Phi)_M$ etc. respectively. Let us consider the natural morphism

$$\phi'\colon T_{M_{\mathrm{reg}}}X \longrightarrow T_{M_{\mathrm{reg}}}X$$

induced by $\phi\colon X \longrightarrow X$, where M_{reg} denotes the set of regular points in M.

Definition 5. *For $x \in M_{\mathrm{reg}}$, we set*

$$\mathrm{Ev}_x := \{\textit{the eigenvalues of } \phi'_x \colon (T_{M_{\mathrm{reg}}}X)_x \longrightarrow (T_{M_{\mathrm{reg}}}X)_x\} \subset \mathbb{C}.$$

Theorem 2. *Assume the following conditions:*

(i) $\mathrm{supp}(F) \cap M \subset M_{\mathrm{reg}}$ *is compact.*

(ii) $1 \notin \mathrm{Ev}_x$ *for any* $x \in \mathrm{supp}(F) \cap M$.

(iii) $\mathrm{Ev}_x \cap \mathbb{R}_{>1} = \emptyset$ *for any* $x \in \mathrm{supp}(F) \cap M$

(when X, ϕ and F are all complex analytic, (iii) *is not necessary). Then* (2) *holds.*

Proof. Consider the specialization

$$\nu_{M_{\mathrm{reg}}}(F) \in \mathbf{D}^b_{\mathbb{R}-c}(T_{M_{\mathrm{reg}}}X)$$

of F along M_{reg} and the natural morphism

$$\widetilde{\Phi} \colon \phi'^{-1}\nu_{M_{\mathrm{reg}}}(F) \longrightarrow \nu_{M_{\mathrm{reg}}}(F).$$

Then our proof is similar to that of [11, Proposition 9.6.11 and 9.6.12]. We also use some results in Section . For details, see [15]. □

As a very special case of Theorem 2, we obtain the following Lefschetz fixed point formula over singular varieties.

Corollary 1. *Let X be a complex manifold, $\phi \colon X \longrightarrow X$ a holomorphic map and $V \subset X$ a ϕ-invariant compact analytic subset. Assume that for the fixed point set $M = \{x \in X \mid \phi(x) = x\} \subset X$ the following conditions are satisfied.*

(i) $V \cap M \subset M_{\mathrm{reg}}$,

(ii) $1 \notin \mathrm{Ev}_x$ *for any* $x \in V \cap M$.

Then we have

$$\mathrm{tr}(\phi|_V) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(V;\mathbb{C}_V) \overset{(\phi|_V)^*}{\longrightarrow} H^j(V;\mathbb{C}_V)\} = \chi(V \cap M).$$

Example 2. *Let $G_n = SL_n(\mathbb{C})$ and let $B_n \subset G_n$ be the Borel subgroup of G_n consisting of upper triangular matrices. Then the homogeneous space $X = G_n/B_n$ is the flag manifold. Take an element*

$$g = \mathrm{diag}(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1\text{-times}},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2\text{-times}},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{n_k\text{-times}})$$

$(n = n_1 + \cdots + n_k)$ in B_n such that $\lambda_i \neq \lambda_j$ for any $i \neq j$, where $\mathrm{diag}(\cdots)$ denotes a diagonal matrix. Let $\phi \colon X \longrightarrow X$ be the left action $l_g \colon X \xrightarrow{\sim} X$ by $g \in B_n \subset G_n$. Then it is easy to see that the fixed point set M of ϕ is a smooth complex submanifold of X. More precisely, M is isomorphic to the disjoint union of $\dfrac{n!}{n_1!\cdots n_k!}$ copies of the product of smaller flag manifolds $G_{n_1}/B_{n_1}\times\cdots\times G_{n_k}/B_{n_k}$. Therefore the assumptions of Corollary 1 are satisfied for any ϕ-invariant analytic subset V of X, if $1 \notin \mathrm{Ev}_x$ for any $x \in M$ (we expect this is always true). Since $g \in B_n$, as a ϕ-invariant analytic subset V we can take any Schubert variety in X.

Example 3. *Let us consider a special case of Example 2 above. Let $X = G_3/B_3$ be the flag manifold consisting of full flags in $\mathbb{C}^3$ and $\phi = l_g \colon X \xrightarrow{\sim} X$ the left action by the element*

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \in B_3 \subset G_3,$$

where $\alpha \neq \beta$ are non-zero complex numbers. In this case, the fixed point set $M \subset X$ of ϕ is the disjoint union of 3-copies of $\mathbb{CP}^1$'s. Let $X = \bigsqcup_{\sigma \in \mathfrak{S}_3} B_3 \sigma B_3 = \bigsqcup_{\sigma \in \mathfrak{S}_3} X_\sigma$ be the Bruhat decomposition of $X = G_3/B_3$. Here an element σ of the symmetric group $\mathfrak{S}_3$ is identified with the matrix $(\delta_{i,\sigma(j)})_{1 \leq i,j \leq 3} \in G_3$ (see e.g. [8] for the detail of this subject), where δ_{ij} is Kronecker's delta. In this case, $X_{(1,3)}$ is the unique open dense Schubert cell in X. Set $V = X \setminus X_{(1,3)} = \bigsqcup_{\sigma \neq (1,3)} X_\sigma$. Then V is a ϕ-invariant analytic subset of X and we can check that the assumptions of Corollary 1 are satisfied.

Example 4. *Let $N \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice, Δ a complete fan in $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$ and $X(\Delta)$ the toric variety associated with Δ. Denote the open dense algebraic torus of $X(\Delta)$ by $T \simeq (\mathbb{C}^*)^n$. Since T acts on $X(\Delta)$ itself, for each $g \in T$ we can consider the left action $\phi := l_g \colon X(\Delta) \longrightarrow X(\Delta)$ by g. Assume that $X(\Delta)$ is smooth. Then for any $g \in T$ the fixed point set M of $\phi = l_g$ is smooth and we can easily verify that $1 \notin \mathrm{Ev}_x$ for any $x \in M$. We assume also that there exists a Laurent polynomial $f \colon T \simeq (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ which satisfies the condition*

$$\exists C \in \mathbb{C} \quad s.t. \quad f(a_1 x_1, \cdots, a_n x_n) = C \cdot f(x_1, \cdots, x_n),$$

where we set $g = (a_1, \cdots, a_n) \in T \simeq (\mathbb{C}^*)^n$ (for example, if $g = (\alpha^{m_1}, \alpha^{m_2}, \cdots, \alpha^{m_n})$ we can take f to be any quasi-homogeneous polynomial of weight $(m_1, m_2, \cdots, m_n)$). Then the toric hypersurface $V = \overline{\{x \in T \mid f(x) = 0\}} \subset X(\Delta)$ associated with f is invariant by $\phi = l_g \colon X(\Delta) \longrightarrow X(\Delta)$. Hence we can use Corollary 1 to conclude that the Lefschetz number $\mathrm{tr}(\phi|_V)$ of $\phi|_V \colon V \longrightarrow V$ is $\chi(V \cap M)$.

3. Microlocal Study of Lefschetz Fixed Point Formulas

3.1. Definition of Lefschetz Cycles

In this section, we construct certain Lagrangian cycles which encode the local contributions discussed in previous sections into topological objects. We inherit the notations in Sections 1 and 2. For the sake of simplicity, we assume that the fixed point set $M = \{x \in X \mid \phi(x) = x\}$ of $\phi \colon X \longrightarrow X$ is a submanifold of X. However, here we do not assume that M is connected. We also assume that the diagonal set $\Delta_X \simeq X \subset X \times X$ intersects with the graph $\Gamma_\phi = \{(\phi(x), x) \in X \times X \mid x \in X\}$ of ϕ cleanly along M in $X \times X$. Note that the last condition is equivalent to the one: $1 \notin \mathrm{Ev}_x$ for any $x \in M$. Identifying Γ_ϕ with X by the second projection $X \times X \longrightarrow X$, we obtain a natural identification $M = \Gamma_\phi \cap \Delta_X$. We also identify $T_{\Delta_X}(X \times X)$ (resp. $T^*_{\Delta_X}(X \times X)$) with TX (resp. T^*X) by the first projection $T(X \times X) \simeq TX \times TX \longrightarrow TX$ (resp. $T^*(X \times X) \simeq T^*X \times T^*X \longrightarrow T^*X$) as usual. Then, under the above assumptions, we see that the natural morphism

$$T_M \Gamma_\phi \longrightarrow T_{\Delta_X}(X \times X) \simeq TX$$

induced by the inclusion map $\Gamma_\phi \hookrightarrow X \times X$ is injective and the image of this morphism is a subbundle of $M \times_X TX$. We denote this vector bundle over M by $\mathcal{E}$. The following lemma will be obvious.

Lemma 1. *The subset $T^*_{\Gamma_\phi}(X \times X) \cap T^*_{\Delta_X}(X \times X)$ of $(\Gamma_\phi \cap \Delta_X) \times_{\Delta_X} T^*_{\Delta_X}(X \times X) \simeq M \times_X T^*X$ is naturally identified with the subset of $M \times_X T^*X$ consisting of covectors which are orthogonal to the vectors in $\mathcal{E} \subset M \times_X TX$ with respect to the natural pairing $(M \times_X TX) \times (M \times_X T^*X) \longrightarrow \mathbb{R}$.*

By this lemma, we see that $T^*_{\Gamma_\phi}(X \times X) \cap T^*_{\Delta_X}(X \times X)$ is a subbundle of $M \times_X T^*X$. We denote it by $\mathcal{F}$ and call it the Lefschetz bundle associated with $\phi \colon X \longrightarrow X$. The Lefschetz bundles satisfy the following nice property.

Proposition 2. *The natural surjective morphism $\rho \colon M \times_X T^*X \twoheadrightarrow T^*M$ induces an isomorphism $\mathcal{F} \xrightarrow{\sim} T^*M$.*

From now on, by Proposition 2 we shall always identify the Lefschetz bundle $\mathcal{F}$ with T^*M. Now let F be an object of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ and $\Phi\colon \phi^{-1}F \longrightarrow F$ a morphism in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. To the pair (F, Φ), we can associate a conic Lagrangian cycle in the Lefschetz bundle $\mathcal{F} \simeq T^*M$ as follows. Let

$$\mu_{\Delta_X}\colon \mathbf{D}^b(X \times X) \longrightarrow \mathbf{D}^b(T^*_{\Delta_X}(X \times X))$$

be the microlocalization functor along Δ_X. Recall that the micro-support $\mathrm{SS}(F)$ of F is a closed conic subanalytic Lagrangian subset of T^*X and the support of $\mu_{\Delta_X}(F \boxtimes \mathrm{D}F)$ is contained in $\mathrm{SS}(F) \subset T^*X \simeq T^*_{\Delta_X}(X \times X)$. Let $\delta_X\colon X \longrightarrow X \times X$ be the diagonal embedding and $h\colon X \longrightarrow X \times X$ a morphism defined by $x \longmapsto (\phi(x), x)$. We denote by $\pi_M\colon \mathcal{F} \simeq T^*M \longrightarrow M$ the projection. Then we have a chain of natural morphisms:

$$\begin{aligned}
R\mathrm{Hom}_{\mathbb{C}_X}(F, F) &\simeq R\Gamma(X; \delta_X^!(F \boxtimes \mathrm{D}F)) \\
&\simeq R\Gamma_{\mathrm{SS}(F)}(T^*X; \mu_{\Delta_X}(F \boxtimes \mathrm{D}F)) \\
&\longrightarrow R\Gamma_{\mathrm{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*h^{-1}(F \boxtimes \mathrm{D}F))) \\
&\simeq R\Gamma_{\mathrm{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*(\phi^{-1}F \otimes \mathrm{D}F))) \\
&\xrightarrow{\Phi} R\Gamma_{\mathrm{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*(F \otimes \mathrm{D}F))) \\
&\longrightarrow R\Gamma_{\mathrm{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*\omega_X)) \\
&\simeq R\Gamma_{\mathrm{SS}(F)\cap\mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M),
\end{aligned}$$

where we used the isomorphism $\mu_{\Delta_X}(h_*\omega_X)|_{\mathcal{F}} \simeq \pi_M^{-1}\omega_M$ in the last step. Taking the 0-th hypercohomology groups of both sides, we obtain a morphism

$$\mathrm{Hom}_{\mathbf{D}^b(X)}(F, F) \longrightarrow H^0_{\mathrm{SS}(F)\cap\mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M). \tag{3}$$

Definition 6. *We denote by $LC(F, \Phi)$ the image of $\mathrm{id}_F \in \mathrm{Hom}_{\mathbf{D}^b(X)}(F, F)$ in $H^0_{\mathrm{SS}(F)\cap\mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M)$ by (3) (since $\mathrm{SS}(F)\cap\mathcal{F}$ is contained in a closed conic subanalytic Lagrangian subset of $\mathcal{F} \simeq T^*M$, $LC(F, \Phi)$ is a Lagrangian cycle in $\mathcal{F} \simeq T^*M$). We call $LC(F, \Phi)$ the Lefschetz cycle associated with the pair (F, Φ).*

Note that a similar construction of microlocal Lefschetz classes was also given by Guillermou [7]. The difference from his construction is that we explicitly realize such microlocal characteristic classes as geometric objects in the cotangent bundle T^*M. Note also that if $\phi = \mathrm{id}_X$, $M = X$ and $\Phi = \mathrm{id}_F$, our Lefschetz cycle $LC(F, \Phi)_M$ coincides with the characteristic cycle $CC(F)$ of F introduced by Kashiwara [9] (for the applications of characteristic cycles to projective duality, see [4], [13], [14] etc.) By our Lefschetz cycles, we can generalize almost all nice properties of characteristic cycles into more general situations (see the subsequent subsections).

As a basic property of Lefschetz cycles, we have the following homotopy invariance. Let $I = [0, 1]$ and let $\phi\colon X \times I \longrightarrow X$ be the restriction of a morphism of real analytic manifolds $X \times \mathbb{R} \longrightarrow X$. For $t \in I$, let $i_t\colon X \hookrightarrow X \times I$ be the injection defined by $x \longmapsto (x, t)$ and set $\phi_t := \phi \circ i_t\colon X \longrightarrow X$. Assume that the fixed point set of ϕ_t in X is smooth and does not depend on $t \in I$. We denote this fixed point set by M. Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ and consider a morphism $\Phi\colon \phi^{-1}F \longrightarrow p^{-1}F$ in $\mathbf{D}^b_{\mathbb{R}-c}(X \times I)$, where $p\colon X \times I \longrightarrow X$ is the projection. We set

$$\Phi_t := \Phi|_{X \times \{t\}}\colon \phi_t^{-1}F \longrightarrow F$$

for $t \in I$. We denote the Lefschetz bundle associated with ϕ_t by $\mathcal{F}_t \simeq T^*M$.

Proposition 3. *Assume that $\mathrm{supp}(F) \cap M$ is compact and $\mathcal{F}_t$ does not depend on $t \in I$. Then the Lefschetz cycle $LC(F, \Phi_t) \in H^0_{\mathrm{SS}(F)\cap T^*M}(T^*M; \pi_M^{-1}\omega_M)$ does not depend on $t \in I$.*

3.2. Microlocal Index Formula for Local Contributions

In this subsection, we introduce a microlocal index theorem for local contributions. Our theorem is a natural generalization of Kashiwara's microlocal index theorem for characteristic cycles (see

[11, Theorem 9.5.3]). Let $M = \bigsqcup_{i\in I} M_i$ be the decomposition of M into connected components. Set $\mathcal{F}_i := M_i \times_M \mathcal{F}$. Then we get a decomposition $\mathcal{F} = \bigsqcup_{i\in I} \mathcal{F}_i \simeq \bigsqcup_{i\in I} T^*M_i$ of $\mathcal{F}$. By the direct sum decomposition

$$H^0_{\mathrm{SS}(F)\cap\mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M) \simeq \bigoplus_{i\in I} H^0_{\mathrm{SS}(F)\cap\mathcal{F}_i}(\mathcal{F}_i; \pi_{M_i}^{-1}\omega_{M_i}) \simeq \bigoplus_{i\in I} H^{\dim M_i}_{\mathrm{SS}(F)\cap\mathcal{F}_i}(\mathcal{F}_i; \pi_{M_i}^{-1} or_{M_i}),$$

we obtain a decomposition

$$LC(F,\Phi) = \sum_{i\in I} LC(F,\Phi)_{M_i}$$

of $LC(F,\Phi)$, where $LC(F,\Phi)_{M_i} \in H^{\dim M_i}_{\mathrm{SS}(F)\cap\mathcal{F}_i}(\mathcal{F}_i; \pi_{M_i}^{-1} or_{M_i})$. Now let us fix a fixed point component M_i. We shall show how the local contribution $c(F,\Phi)_{M_i} \in \mathbb{C}$ of (F,Φ) from M_i can be expressed by $LC(F,\Phi)_{M_i}$. In order to state our results, for the sake of simplicity, we denote M_i, $\mathcal{F}_i$, $LC(F,\Phi)_{M_i}$, $c(F,\Phi)_{M_i}$ simply by M, $\mathcal{F}$, $LC(F,\Phi)$, $c(F,\Phi)$ respectively. Recall that to any continuous section $\sigma\colon M \longrightarrow \mathcal{F} \simeq T^*M$ of the vector bundle $\mathcal{F}$, we can associate a cycle $[\sigma] \in H^0_{\sigma(M)}(T^*M; \pi_M^!(\mathbb{C}_M))$ by the isomorphism $H^0_{\sigma(M)}(T^*M; \pi_M^!\mathbb{C}_M) \simeq H^0(M; (\pi_M\circ\sigma)^!\mathbb{C}_M) \simeq H^0(M;\mathbb{C}_M)$ and $1 \in H^0(M;\mathbb{C}_M)$ (see [11, Definition 9.3.5]). If $\sigma(M)\cap \mathrm{supp}(LC(F,\Phi))$ is compact, we can define the intersection number $\sharp([\sigma]\cap LC(F,\Phi))$ of $[\sigma]$ with $LC(F,\Phi)$ as the image of $[\sigma]\otimes LC(F,\Phi)$ by the chain of natural morphisms

$$\begin{aligned} H^0_{\sigma(M)}(\mathcal{F}; \pi_M^!\mathbb{C}_M) \otimes H^0_{\mathrm{supp}(LC(F,\Phi))}(\mathcal{F}; \pi_M^{-1}\omega_M) &\longrightarrow H^0_{\sigma(M)\cap\mathrm{supp}(LC(F,\Phi))}(\mathcal{F}; \omega_{\mathcal{F}}) \\ &\xrightarrow{\int_{\mathcal{F}}} \mathbb{C}. \end{aligned}$$

Theorem 3. *Assume that* $\mathrm{supp}(F)\cap M$ *is compact. Then for any continuous section* $\sigma\colon M \longrightarrow \mathcal{F} \simeq T^*M$ *of* $\mathcal{F}$*, we have*

$$c(F,\Phi) = \sharp([\sigma]\cap LC(F,\Phi)).$$

As an application of Theorem 3, we shall give a useful formula which enables us to describe the Lefschetz cycle $LC(F,\Phi)$ explicitly in the special case where $\phi\colon X \longrightarrow X$ is the identity map of X and $M = X$. For this purpose, until the end of this subsection, we shall consider the situation where $\phi = \mathrm{id}_X$, $M = X$ and $\Phi\colon F \longrightarrow F$ is an endomorphism of $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$. In this case, $LC(F,\Phi)$ is a Lagrangian cycle in T^*X. For a real-valued real analytic function $\varphi\colon X \longrightarrow \mathbb{R}$ on X, we define a section $\sigma_\varphi\colon X \longrightarrow T^*X$ of T^*X by $\sigma_\varphi(x) := (x; d\varphi(x))$ $(x\in X)$ and set

$$\Lambda_\varphi := \sigma_\varphi(X) = \{(x; d\varphi(x)) \mid x \in X\}.$$

Note that Λ_φ is a Lagrangian submanifold of T^*X. Then we have the following analogue of [11, Theorem 9.5.3].

Let $X = \bigsqcup_{\alpha\in A} X_\alpha$ be a μ-stratification (for the definition see [11, Definition 8.3.19]) of X such that

$$\mathrm{supp}(LC(F,\Phi)) \subset \mathrm{SS}(F) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X.$$

Then $\Lambda := \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$ is a closed conic subanalytic Lagrangian subset of T^*X. Moreover there exists an open dense smooth subanalytic subset Λ_0 of Λ whose decomposition $\Lambda_0 = \bigsqcup_{i\in I}\Lambda_i$ into connected components satisfies the condition

"For any $i \in I$, there exists $\alpha_i \in A$ such that $\Lambda_i \subset T^*_{X_{\alpha_i}}X$. "

Definition 7. *For* $i\in I$ *and* $\alpha_i \in A$ *etc. as above, we define a complex number* $m_i \in \mathbb{C}$ *by*

$$m_i := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{tr}\{H^j_{\{\varphi\geq\varphi(x)\}}(F)_x \xrightarrow{\Phi} H^j_{\{\varphi\geq\varphi(x)\}}(F)_x\},$$

where $x \in \pi_X(\Lambda_i) \subset X_{\alpha_i}$ *and the real-valued real analytic function* $\varphi\colon X \longrightarrow \mathbb{R}$ *(defined in an open neighborhood of* x *in* X*) are defined as follows. Take a point* $p\in\Lambda_i$ *and set* $x = \pi_X(p) \in X_{\alpha_i}$*. Then* $\varphi\colon X \longrightarrow \mathbb{R}$ *is a real analytic function which satisfies the following conditions:*

(i) $p = (x; d\varphi(x)) \in \Lambda_i$.

(ii) *The Hessian* $\mathrm{Hess}(\varphi|_{X_{\alpha_i}})$ *of* $\varphi|_{X_{\alpha_i}}$ *is positive definite.*

Theorem 4. *In the situation as above, for any* $i \in I$ *there exists an open neighborhood* U_i *of* Λ_i *in* T^*X *such that*

$$LC(F, \Phi) = m_i \cdot [T^*_{X_{\alpha_i}} X]$$

in U_i.

3.3. Explicit Description of Lefschetz Cycles

In this subsection, we explicitly describe the Lefschetz cycle $LC(F, \Phi)_M$ introduced in Section in many cases. Let M be a possibly singular fixed point component of $\phi\colon X \longrightarrow X$. Throughout this subsection, we assume the condition

$$\text{"}1 \notin \mathrm{Ev}_x \text{ for any } x \in \mathrm{supp}(F) \cap M_{\mathrm{reg}}.\text{"}$$

By making use of some localization theorems for Lefschetz cycles similar to the one in Section 2, we obtain the following explicit description of $LC(F, \Phi)_M$.

Theorem 5. *Let* $x_0 \in M_{\mathrm{reg}}$ *be a point of* M_{reg} *such that*

$$\mathrm{Ev}_{x_0} \cap \mathbb{R}_{\geq 1} = \emptyset. \tag{4}$$

Then we have

$$LC(F, \Phi)_M = LC(F|_M, \Phi|_M)_M$$

in an open neighborhood of $\pi_M^{-1}(x_0)$ *in* T^*M_{reg}.

In the complex case, we have the following stronger result.

Theorem 6. *In the above situation, assume that* X *and* $\phi\colon X \longrightarrow X$ *are complex analytic and* $F \in \mathbf{D}^b_c(X)$ *i.e.* F *is* $\mathbb{C}$*-constructible. Then we have*

$$LC(F, \Phi)_M = LC(F|_M, \Phi|_M)_M$$

globally on T^*M_{reg}.

Combining Theorem 5, 6 with Theorem 4, we thus obtain the explicit description of Lefschetz cycles. By this description, we also obtain the following result.

Corollary 2. *Let* X, ϕ *and* M *be as above and* $F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \xrightarrow{\gamma} +1$ *a distinguished triangle in* $\mathbf{D}^b_{\mathbb{R}-c}(X)$. *Assume that we are given a commutative diagram*

$$\begin{tikzcd}
\phi^{-1}F_1 \arrow[r, "\phi^{-1}\alpha"] \arrow[d, "\Phi_1"] & \phi^{-1}F_2 \arrow[r, "\phi^{-1}\beta"] \arrow[d, "\Phi_2"] & \phi^{-1}F_3 \arrow[r, "\phi^{-1}\gamma"] \arrow[d, "\Phi_3"] & \phi^{-1}F_1[1] \arrow[d, "\Phi_1[1]"] \\
F_1 \arrow[r, "\alpha"] & F_2 \arrow[r, "\beta"] & F_3 \arrow[r, "\gamma"] & F_1[1]
\end{tikzcd}$$

in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. *Then for any* $x_0 \in M_{\mathrm{reg}}$ *such that* $\mathrm{Ev}_{x_0} \cap \mathbb{R}_{\geq 1} = \emptyset$, *we have*

$$LC(F_2, \Phi_2)_M = LC(F_1, \Phi_1)_M + LC(F_3, \Phi_3)_M$$

in an open neighborhood of $\pi_M^{-1}(x_0)$ *in* T^*M_{reg}.

3.4. Direct Image Theorem

From now on, we study functorial properties of our Lefschetz cycles.

Let $f\colon Y \longrightarrow X$ be a morphism of real analytic manifolds.. Assume that we are given two morphisms $\phi_X\colon X \longrightarrow X$ and $\phi_Y\colon Y \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\phi_Y"'] & X \arrow[d, "\phi_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

commutes. Let us take an object G of $\mathbf{D}^b_{\mathbb{R}-c}(Y)$ such that f is proper on $\mathrm{supp}(G)$ and a morphism

$$\Phi_Y\colon \phi_Y^{-1}G \longrightarrow G$$

in $\mathbf{D}^b_{\mathbb{R}-c}(Y)$. Then $Rf_*G \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ and we obtain a natural morphism

$$\Phi_X\colon \phi_X^{-1}Rf_*G \longrightarrow Rf_*G$$

induced by Φ_Y. Our aim in this subsection is to compare the Lefschetz cycle of (G, Φ_Y) with that of (Rf_*G, Φ_X). Let M be a smooth fixed point component of ϕ_X such that $f(\mathrm{supp}(G)) \cap M$ is compact. Also let $\{N_i\}_{i\in I}$ be the set of all fixed point components N_i of ϕ_Y such that $N_i \subset f^{-1}(M)$ and $\mathrm{supp}(G) \cap N_i \neq \emptyset$. Note that I is a finite set by our assumptions. Set $N := \bigsqcup_{i\in I} N_i$ and assume that N is smooth. We also assume that $\Gamma_{\phi_X} \subset X \times X$ (resp. $\Gamma_{\phi_Y} \subset Y \times Y$) intersects with Δ_X in $X \times X$ (resp. Δ_Y in $Y \times Y$) cleanly along M (resp. N) as in previous sections. For the sake of simplicity, we denote $M \times_X \{T^*_{\Gamma_{\phi_X}}(X \times X) \cap T^*_{\Delta_X}(X \times X)\} \simeq T^*M$, $N \times_Y \{T^*_{\Gamma_{\phi_Y}}(Y \times Y) \cap T^*_{\Delta_Y}(Y \times Y)\} \simeq T^*N$ simply by $\mathcal{F}$, $\mathcal{G}$ respectively. Then we obtain two Lefschetz bundles

$$\mathcal{F} \subset T^*_{\Gamma_{\phi_X}}(X \times X) \cap T^*_{\Delta_X}(X \times X),$$
$$\mathcal{G} \subset T^*_{\Gamma_{\phi_Y}}(Y \times Y) \cap T^*_{\Delta_Y}(Y \times Y)$$

and the Lefschetz cycles

$$LC(G, \Phi_Y)_N \in H^0_{\mathrm{SS}(G)\cap\mathcal{G}}(\mathcal{G}; \pi_N^{-1}\omega_N),$$
$$LC(Rf_*G, \Phi_X)_M \in H^0_{\mathrm{SS}(Rf_*G)\cap\mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M)..$$

Note that by setting $\mathcal{G}_i := N_i \times_N \mathcal{G}$ we have the direct sum decompositions $\mathcal{G} = \bigsqcup_{i\in I} \mathcal{G}_i \simeq \bigsqcup_{i\in I} T^*N_i$ and

$$LC(G, \Phi_Y)_N = \sum_{i\in I} LC(G, \Phi_Y)_{N_i},$$

where $LC(G, \Phi_Y)_{N_i} \in H^0_{\mathrm{SS}(G)\cap\mathcal{G}_i}(\mathcal{G}_i; \pi_{N_i}^{-1}\omega_{N_i})$. Now, set $g = f|_N\colon N \longrightarrow M$ and consider the natural morphisms

$$T^*N \xleftarrow{\ {}^tg'\ } N \times_M T^*M \xrightarrow{\ g_\pi\ } T^*M$$

induced by g. Take a closed conic subanalytic Lagrangian subset $\Lambda = \bigsqcup_{i\in I} \Lambda_i$ of $T^*N = \bigsqcup_{i\in I} T^*N_i$ such that $\mathrm{SS}(G) \cap \mathcal{G} \subset \Lambda$ and set $\Lambda' = {}^tg'^{-1}(\Lambda)$ and $\Lambda'' = g_\pi(\Lambda')$. Then there exists a morphism

$$g_*\colon H^0_\Lambda(T^*N; \pi_N^{-1}\omega_N) \longrightarrow H^0_{\Lambda''}(T^*M; \pi_M^{-1}\omega_M)$$

of Lagrangian cycles induced by g (see [11, Proposition 9.3.2 (i)])..

Theorem 7. *In the above situation, we have*

$$LC(Rf_*G, \Phi_X)_M = g_*(LC(G, \Phi_Y)_N)$$

in T^*M. More precisely, for the morphism

$$(g_i)_* \colon H^0_{\Lambda_i}(T^*N_i; \pi_{N_i}^{-1}\omega_{N_i}) \longrightarrow H^0_{\Lambda''}(T^*M; \pi_M^{-1}\omega_M)$$

of Lagrangian cycles induced by $g_i := f|_{N_i} \colon N_i \longrightarrow M$ we have

$$LC(Rf_*G, \Phi_X)_M = \sum_{i \in I} (g_i)_*(LC(G, \Phi_Y)_{N_i}). \tag{5}$$

Applying Theorem 3 to both sides of (5), we obtain

Corollary 3. *For the local contributions $c(G, \Phi_Y)_{N_i}$ and $c(Rf_*G, \Phi_X)_M$, we have*

$$c(Rf_*G, \Phi_X)_M = \sum_{i \in I} c(G, \Phi_Y)_{N_i}.$$

In fact, Corollary 3 can be proved more directly without using Lefschetz cycles. We can even generalize it as follows (see [16]).

Theorem 8 ([16]). *We inherit the notations and the situation as above. However, we do not assume that M (resp. $N = \bigsqcup_{i \in I} N_i$) is smooth nor $1 \notin \mathrm{Ev}_x$ for $x \in M$ (resp. $1 \notin \mathrm{Ev}_y$ for $y \in N$) here. Then we have*

$$c(Rf_*G, \Phi_X)_M = \sum_{i \in I} c(G, \Phi_Y)_{N_i}.$$

3.5. Inverse Image Theorem

In this subsection, we introduce the inverse image theorem for Lefschetz cycles. We inherit the situation treated in Subsection 3.4. However, here M and N are smooth fixed point components of ϕ_X and ϕ_Y respectively satisfying the condition $f(N) \subset M$. We take an object F of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ and a morphism

$$\Phi_X \colon \phi_X^{-1}F \longrightarrow F$$

in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. Then $f^{-1}F \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ and we obtain a natural morphism

$$\Phi_Y \colon \phi_Y^{-1}f^{-1}F \longrightarrow f^{-1}F$$

induced by Φ_X. Assuming the same conditions on ϕ_X, ϕ_Y etc.. and keeping the same notations for $\mathcal{F}$, $\mathcal{G}$ etc. as in Subsection 3.4, we obtain the Lefschetz cycles

$$LC(F, \Phi_X)_M \in H^0_{\mathrm{SS}(F) \cap \mathcal{F}}(\mathcal{F}; \pi_M^{-1}\omega_M),$$
$$LC(f^{-1}F, \Phi_Y)_N \in H^0_{\mathrm{SS}(f^{-1}F) \cap \mathcal{G}}(\mathcal{G}; \pi_N^{-1}\omega_N).$$

Set $g = f|_N \colon N \longrightarrow M$ as before and consider the natural morphisms

$$T^*N \xleftarrow{{}^t g'} N \times_M T^*M \xrightarrow{g_\pi} T^*M$$

induced by g. Take a closed conic subanalytic Lagrangian subset Λ of T^*M such that $\mathrm{SS}(F) \cap \mathcal{F} \subset \Lambda$ and set $\Lambda' = g_\pi^{-1}(\Lambda)$ and $\Lambda'' = {}^t g'(\Lambda')$. If ${}^t g'$ is proper on Λ' (e.g. if f is non-characteristic for F on an open neighborhood of N), then there exists a morphism

$$g^* \colon H^0_\Lambda(T^*M; \pi_M^{-1}\omega_M) \longrightarrow H^0_{\Lambda''}(T^*N; \pi_N^{-1}\omega_N)$$

of Lagrangian cycles induced by g (see [11, Proposition 9.3.2 (ii)]).

Theorem 9. *In the above situation, assume moreover that f is non-characteristic for F on an open neighborhood of N. Then we have*

$$LC(f^{-1}F, \Phi_Y)_N = \mathrm{sgn}(\mathrm{id} - \phi'_X) \cdot \mathrm{sgn}(\mathrm{id} - \phi'_Y) \cdot g^*(LC(F, \Phi_X)_M)$$

in T^*N, where $\mathrm{sgn}(\mathrm{id} - \phi'_X) = \pm 1$ (resp. $\mathrm{sgn}(\mathrm{id} - \phi'_Y) = \pm 1$) is the sign of the determinant of $\mathrm{id} - \phi'_X \colon T_M X \longrightarrow T_M X$ (resp. $\mathrm{id} - \phi'_Y \colon T_N Y \longrightarrow T_N Y$).

As a special case of this theorem, we obtain the following result which drops the condition (4) of Theorem 5.

Corollary 4. *Under the assumptions in Theorem 5, instead of assuming the condition* (4)*, assume that the inclusion map* $i\colon M \hookrightarrow X$ *of the fixed point manifold* M *is non-characteristic for* F*. Then we have*

$$LC(F,\Phi)_M = \mathrm{sgn}(\mathrm{id}-\phi')\cdot LC(F|_M,\Phi|_M)_M$$

in T^*M*. In particular, if moreover* $\mathrm{supp}(F)\cap M$ *is compact, we have*

$$c(F,\Phi)_M = \mathrm{sgn}(\mathrm{id}-\phi')\cdot \mathrm{tr}(F|_M,\Phi|_M).$$

Note that the last term $\mathrm{tr}(F|_M,\Phi|_M)$ in Corollary 4 can be easily computed by Proposition 1. And note that Corollary 4 is not true if we do not assume that $i\colon M \hookrightarrow X$ is non-characteristic for F. See e.g. [11, Example 9.6.18].

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