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On Decomposition of Sub-definite Partial Boolean Functions

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In this article we study Boolean functions with two kinds of indeterminacy. We prove criterion of decomposition of this functions including separating decomposition. As a result we have method that allows to obtain representation of an arbitrary function using superposition of functions that have smaller dimentions.

Keywords: incompletely defined Boolean function, sub-definite partial Boolean function, decomposition, superposition.
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Introduction

In the theory of discrete functions rapidly developing area, is engaged study of functions defined on a finite set A and receiving as their values subsets of A, including $\varnothing$. Such maps are found in the mathematical modeling of information processing, in the case where the set $A = \{0, 1\}$ are incompletely defined Boolean functions. As can be seen, there are two kinds of indeterminacy. For the first type of indeterminacy on the sets on which the function value is not defined, the indeterminacy is understood as the ability to adopt and value 0 and value 1, i.e. image of these sets is the set $\{0, 1\}$. Boolean functions with this kind of indeterminacy are considered, for example, in [1]. The second type of indeterminacy are associated with the empty set, typically means taboo data and studied, for example, in [2].

In this paper we consider incompletely defined Boolean functions with two kinds of indeterminacy, following [3], we call them sub-definite partial Boolean functions.

The problem of representation of an arbitrary sub-definite partial Boolean function by the functions of lower dimension is very important. We proved criterion of decomposition sub-definite partial Boolean functions, including separating decomposition, which generalizes the criterion of the functional separability of Boolean functions by G. N. Povarov [4] and provides a method of obtaining representations sub-definite partial Boolean functions by the functions of lower dimension.

The work [5] is dedicated to finding of repetition-free representations of sub-definite partial Boolean functions in a special basic set. We note that the obtaining of the results of [5] is greatly simplified by the criterion of separating decomposition. Moreover, our method can be used to construct algorithms of repetition-free representations of sub-definite partial Boolean functions in other basic sets.

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1. Basic concepts and definitions

We preface the description of the main results with the needed definitions and notation. We note that the terminology which used to sub-definite partial Boolean functions, completely preserved from the theory of Boolean functions, which can be seen in [6]. We use the following notation: variables are denoted by the symbols x, y, u, v, w, maybe with subscripts; constants are denoted by the symbols α, β, σ, γ, maybe with subscripts; the symbol $\tilde{x}$ denotes the tuple $(x_1, \dots, x_n)$; $|\tilde{x}|$ is the length of a tuple $\tilde{x}$.

Let $|A|$ is the power of set A, 2^A is the set of all subsets of A, $E_2 = \{0, 1\}$. We define the following sets of functions:

$P_{2,n}^{\bar{*}} = \{f | f : E_2^n \to 2^{E_2}\}, P_2^{\bar{*}} = \bigcup\limits_n P_{2,n}^{\bar{*}}$,

$P_{2,n} = \{f | f \in P_{2,n}^{\bar{*}}$ and $|f(\tilde{\alpha})| = 1$ for every $\tilde{\alpha} \in E_2^n\}, P_2 = \bigcup\limits_n P_{2,n}$.

Functions from P_2 are called Boolean functions, and functions from $P_2^{\bar{*}}$ are called sub-definite partial Boolean functions. Below sub-definite partial Boolean functions are simply called functions.

By definition we believe that the superposition

$$f(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)),$$

where f, f_1, ..., $f_n \in P_2^{\bar{*}}$, represents some function $g(x_1, \dots, x_m)$, if for every $(\alpha_1, \dots, \alpha_m) \in E_2^m$

$$g(\alpha_1, \dots, \alpha_m) = \begin{cases} \varnothing, \text{ if } f_i(\alpha_1, \dots, \alpha_m) = \varnothing \text{ for some } i \in \{1, \dots, m\}; \\ \bigcup\limits_{\beta_i \in f_i(\alpha_1, \dots, \alpha_m)} f(\beta_1, \dots, \beta_n), \text{ otherwise.} \end{cases}$$

The function obtained from $f(x_1, ..., x_n)$ by the substitution of a constant $\sigma \in \{0, 1\}$ for a variable x_i is called the remainder function and is denoted $f_{x_i}^{\sigma}$. This definition is extended to a subset of variables by induction.

For simplicity we will use the following code: $\varnothing \leftrightarrow *$, $\{0\} \leftrightarrow 0$, $\{1\} \leftrightarrow 1$, $\{0, 1\} \leftrightarrow 2$. The function which on all tuples is equal to $*$ will be denoted by $*$.

For arbitrary n-ary functions f и g we define function $f \cup g$ in the following way:

$$(f \cup g)(\alpha_1, \dots, \alpha_n) = f(\alpha_1, \dots, \alpha_n) \cup g(\alpha_1, \dots, \alpha_n)$$

for an arbitrary tuple $(\alpha_1, \dots, \alpha_n)$.

Function f has *decomposition* by partition of set of variables on $\tilde{u}, \tilde{v}, \tilde{w}$, if there exist functions h и g such that holds

$$f(\tilde{u}, \tilde{v}, \tilde{w}) = h(\tilde{u}, \tilde{w}, g(\tilde{u}, \tilde{v})). \tag{1}$$

If $\tilde{u} = \varnothing$, then this decomposition is called *separating.*

2. The main result

In this section we prove necessary and sufficient condition of existence of decomposition and also separating decomposition for an arbitrary function.

Theorem 1. *An arbitrary function f has decomposition by partition of set of variables on $\tilde{u}, \tilde{v}, \tilde{w}$ if and only if for an arbitrary tuple $\tilde{\alpha}$ ($|\tilde{\alpha}| = |\tilde{u}|$) there exist no more than four different remainder* *functions of* $f_{\tilde{u}}^{\tilde{\alpha}}$ *for variables* $\tilde{v}$*, and each of remainder functions is equal to or* $*$*, or some function* f_0*, or some function* f_1*, or* $f_0 \cup f_1$*.*

Proof. Necessity. Since function f has decomposition, then

$$f(\tilde{u}, \tilde{v}, \tilde{w}) = h(\tilde{u}, \tilde{w}, g(\tilde{u}, \tilde{v})).$$

For arbitrary tuples $\tilde{\alpha}$ and $\tilde{\beta}$ we have

$$f(\tilde{\alpha}, \tilde{\beta}, \tilde{w}) = h(\tilde{\alpha}, \tilde{w}, g(\tilde{\alpha}, \tilde{\beta})).$$

Because of $g(\tilde{\alpha}, \tilde{\beta}) \in \{0, 1, *, 2\}$, the remainder function $f(\tilde{\alpha}, \tilde{\beta}, \tilde{w})$ is equal to or $*$, or $h(\tilde{\alpha}, \tilde{w}, 0)$, or $h(\tilde{\alpha}, \tilde{w}, 1)$, or $h(\tilde{\alpha}, \tilde{w}, 2) = h(\tilde{\alpha}, \tilde{w}, 0) \cup h(\tilde{\alpha}, \tilde{w}, 1)$.

Sufficiency. We define functions $g(\tilde{u}, \tilde{v})$ and $h(\tilde{u}, \tilde{w}, y)$. For arbitrary tuples $\tilde{\alpha}$ and $\tilde{\beta}$

$$g(\tilde{\alpha}, \tilde{\beta}) = \begin{cases} *, \text{ if } f(\tilde{\alpha}, \tilde{\beta}, \tilde{w}) = *; \\ 0, \text{ if } f(\tilde{\alpha}, \tilde{\beta}, \tilde{w}) = f_0(\tilde{w}); \\ 1, \text{ if } f(\tilde{\alpha}, \tilde{\beta}, \tilde{w}) = f_1(\tilde{w}); \\ 2, \text{ if } f(\tilde{\alpha}, \tilde{\beta}, \tilde{w}) = f_0(\tilde{w}) \cup f_1(\tilde{w}). \end{cases}$$

and

$$h(\tilde{\alpha}, \tilde{w}, y) = \begin{cases} f_0(\tilde{w}), \text{ if } y = 0; \\ f_1(\tilde{w}), \text{ if } y = 1. \end{cases}$$

We show that for such functions g and h the equality (1) holds. We consider $h(\tilde{\alpha}, \tilde{\gamma}, g(\tilde{\alpha}, \tilde{\beta}))$ for arbitrary $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$.

If $g(\tilde{\alpha}, \tilde{\beta}) = *$, then $h(\tilde{\alpha}, \tilde{\gamma}, g(\tilde{\alpha}, \tilde{\beta})) = * = f(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

If $g(\tilde{\alpha}, \tilde{\beta}) = 0$, then $h(\tilde{\alpha}, \tilde{\gamma}, g(\tilde{\alpha}, \tilde{\beta})) = f_0(\tilde{\gamma}) = f(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

If $g(\tilde{\alpha}, \tilde{\beta}) = 1$, then $h(\tilde{\alpha}, \tilde{\gamma}, g(\tilde{\alpha}, \tilde{\beta})) = f_1(\tilde{\gamma}) = f(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$.

If $g(\tilde{\alpha}, \tilde{\beta}) = 2$, then $h(\tilde{\alpha}, \tilde{\gamma}, g(\tilde{\alpha}, \tilde{\beta})) = f_0(\tilde{\gamma}) \cup f_1(\tilde{\gamma}) = f(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$. □

Corollary 1 (Criterion of separating decomposition). *An arbitrary function* f *has separating decomposition by partition of set of variables on* $\tilde{v}, \tilde{w}$ *if and only if there exist no more than four different remainder functions of* f *for variables* $\tilde{v}$*, and each of remainder functions is equal to or* $*$*, or some function* f_0*, or some function* f_1*, or* $f_0 \cup f_1$*.*

Proof follows from proof of theorem when $\tilde{u} = \varnothing$. □

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О декомпозиции недоопределенных частичных булевых функций

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В статье рассматриваются булевы функции с двумя видами неопределенности. Доказан критерий декомпозиции, в том числе разделительной декомпозиции таких функций, который дает метод, позволяющий получать представление произвольной функции с помощью суперпозиции функций меньших размерностей.

Ключевые слова: не всюду определенная булева функция, недоопределенная частичная булева функция, декомпозиция, суперпозиция.