Existence Criteria in Some Extremum Problems Involving Eigenvalues of Elliptic Operators

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Some existence criteria for a certain class of extremum problems involving eigenvalues of linear elliptic boundary-value problems (including ones in the form of variational inequalities) are proved. The approach applied admits an extension to the case of extremum problems associated with eigenvalues of nonlinear boundary-value problems. Some applications to optimal structural design and comparisons with results in the literature are given.

Keywords: eigenvalue optimization problem, elliptic boundary-value problem, variational inequality, existence theorem, optimal structural design.

Introduction

Extremum problems involving eigenvalues of elliptic boundary-value problems are of great interest and value. A large number of such problems often arise in optimal structural design (see [1, 2] for more details). For example, in order to widen a resonance-free frequency interval of some structure it is sufficient to maximize either its first natural frequency or the difference between the corresponding adjacent frequencies. One of the most important characteristics of a structure is also the critical load under which the structure loses stability. Therefore, it is interesting to maximize this characteristic of the structure. The frequencies of the natural oscillations of a structure and the critical load that causes buckling of the structure correspond to eigenvalues of appropriate boundary-value problems. Thus, there exists a class of extremal problems for eigenvalue functionals in optimal structural design.

Optimization problems for eigenvalues of elliptic operators have been considered by many authors (see [1–9]). For surveys on such problems we refer the reader to [1–3]. Such problems, under the assumption that admissible controls form a weakly compact set of a Sobolev space, were considered in [2, 4]. Let us advance some arguments in favour of consideration of broader sets of admissible controls for such problems. Firstly, the condition of uniform boundness of the first-order weak derivatives of functions that belong to Sobolev spaces leads to using additional techniques, such as those utilizing penalty methods, to implement numerical procedures to derive optimal solutions to such problems. Secondly, the controls corresponding to such functions are often unnatural for applications. Finally, classes of admissible controls arising in many applications whose elements are essentially bounded measurable functions are weak* compact without any artificial supplementary constraints. Let us illustrate the essence of the second argument by means of an example. For a thin cylindrical rod clamped at both ends and having constant flexural rigidity and a given total mass, consider the problem of determining optimal density distributions which yield the highest possible value of the first natural frequency of the

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rod. It is well known (see [3, 5]) that there exists a unique optimal solution to the problem which is a concrete piecewise constant function. Note that, in general, making of a rod having a continuous density distribution is complicated.

Some existence results for extremal eigenvalue problems in the case of composite membranes were presented in [6]. The problem of optimal design of a column against buckling under various boundary conditions was studied in [7]. Such a problem without any positive lower bound for design functions in the case of columns clamped at both ends was considered in [8]. The problem of maximization of the first natural frequency of a clamped thin isotropic plate was investigated in [9]. In that paper instead of proving existence of an optimal solution to the original problem, a family of auxiliary regularized optimization problems depending on a parameter was introduced and existence of an optimal solution to each regularized problem was established. Emphasize that the problems considered in [5–8] are optimal control problems for which only one coefficient depends on an appropriate control. Thus, it is of interest to obtain some general existence criteria for a certain class of extremum problems involving eigenvalues of elliptic operators which contains as many as possible applied problems, including optimal control problems for which several coefficients depend on controls.

In this paper, without using any regularization techniques we prove such existence criteria for extremum problems associated with functionals defined on weak* compact sets.

The structure of the paper is as follows. In Section 1 we formulate some optimization problems for eigenvalues of elliptic boundary-value problems, including ones in the form of variational inequalities, present basic assumptions. Some existence criteria are proved in Section 2. The principal tools of the proofs of the criteria are variational properties of eigenvalues and semicontinuity of integral functionals. Note that the methods used in this paper are quite different from those used in the above-mentioned studies. Moreover, though the main results are proved for the linear case, ones can be easily extended to the case of nonlinear eigenvalue problems for elliptic systems provided that there exists a variational characterization for eigenvalues of such problems. As it turns out, the approach presented in this paper can be directly applied to many concrete problems in optimal structural design, including problems in which some natural frequencies of a structure and the critical load, under which the structure loses stability, are a part of constraints. In order to demonstrate this, we give some interesting applications in Section 3.

1.   Optimization problems

Let H be a Hilbert space. For the sake of convenience we denote the fact that M is a closed subspace of H and is equipped with the same scalar product by $M \leqslant H$. Let Ω be a non-empty bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$. Next, let s, m and l be natural numbers such that $m \leqslant l < s$. Consider linear spaces W, $\mathcal{W}$ and V such that

$$V \leqslant H^s(\Omega), \qquad \mathcal{W} \leqslant H^l(\Omega), \qquad W \leqslant H^m(\Omega), \qquad C_0^\infty(\bar{\Omega}) \subset V \subset \mathcal{W} \subset W.$$

Here $H^j(\Omega)$ denotes the Sobolev space equipped with the scalar product

$$\langle y, z \rangle_j = \sum_{|\alpha| \leq j} \int_\Omega \partial^\alpha y \, \partial^\alpha z \, dx, \qquad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d, \qquad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Furthermore, we assume that

$$\text{the properties of } \Omega \text{ cause the compactness of the imbedding operator of } V \text{ to } \mathcal{W}. \qquad (1)$$

Next, let U be a non-empty bounded set of $L_r^\infty(\Omega)$, i.e.

$$\forall u = (u_1, \dots, u_r) \in U \Rightarrow -\infty < \check{u}_i \leqslant u(x) \leqslant \hat{u}_i < +\infty \ \text{ a.e. in } \ \Omega.$$

We denote the norms on V, W and the standard norm on $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ by $\|\cdot\|_V$, $\|\cdot\|_W$ and $\|\cdot\|_p$, respectively. By ϑ we denote the neutral element of V. Let $\mathcal{G} = \prod_{i=1}^{r} [\check{u}_i, \hat{u}_i]$.

For any $u \in U$, consider two bilinear forms $\mathcal{A}_u : V \times V \to \mathbb{R}$ and $\mathcal{B}_u : W \times W \to \mathbb{R}$, which are defined as follows:

$$\mathcal{A}_u(y, z) = \sum_{|\alpha|, |\beta| \leqslant s} \int_\Omega a_{\alpha\beta}\left(x, u(x)\right) \partial^\alpha y(x)\, \partial^\beta z(x)\, dx,$$

$$\mathcal{B}_u(y, z) = \sum_{|\alpha|, |\beta| \leqslant m} \int_\Omega b_{\alpha\beta}\left(x, u(x)\right) \partial^\alpha y(x)\, \partial^\beta z(x)\, dx.$$

Here $a_{\alpha\beta}(\cdot, \cdot)$ and $b_{\alpha\beta}(\cdot, \cdot)$ are functions defined on $\Omega \times \mathcal{G}$ such that

$$(x, \xi) \mapsto a_{\alpha\beta}(x, \xi) : \Omega \times \mathcal{G} \to [\check{a}_{\alpha\beta}, \hat{a}_{\alpha\beta}], \quad -\infty < \check{a}_{\alpha\beta} \leqslant \hat{a}_{\alpha\beta} < +\infty, \quad a_{\alpha\beta} = a_{\beta\alpha},$$

$$(x, \xi) \mapsto b_{\alpha\beta}(x, \xi) : \Omega \times \mathcal{G} \to [\check{b}_{\alpha\beta}, \hat{b}_{\alpha\beta}], \quad -\infty < \check{b}_{\alpha\beta} \leqslant \hat{b}_{\alpha\beta} < +\infty, \quad b_{\alpha\beta} = b_{\beta\alpha}.$$

Furthermore, we assume that $a_{\alpha\beta}(\cdot, \cdot)$ and $b_{\alpha\beta}(\cdot, \cdot)$ satisfy the Carathéodory conditions. Recall that a function $f : \Omega \times \mathcal{G} \to \mathbb{R}$ is said to satisfy the Carathéodory conditions if $f(\cdot, \xi)$ is measurable for each $\xi \in \mathcal{G}$, and $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$. Since $a_{\alpha\beta}(\cdot, \cdot)$ and $b_{\alpha\beta}(\cdot, \cdot)$ satisfy the Carathéodory conditions, we have that for any $u \in U$ maps $x \mapsto a_{\alpha\beta}(x, u(x))$, $x \mapsto b_{\alpha\beta}(x, u(x))$ are measurable and in $L^\infty(\Omega)$.

It is pretty straightforward that $\mathcal{A}_u(\cdot, \cdot)$ and $\mathcal{B}_u(\cdot, \cdot)$ are symmetric:

$$\mathcal{A}_u(y_1, z_1) = \mathcal{A}_u(z_1, y_1), \quad y_1, z_1 \in V, \qquad \mathcal{B}_u(y_2, z_2) = \mathcal{B}_u(z_2, y_2), \quad y_2, z_2 \in W. \tag{2}$$

Moreover, $\mathcal{A}_u(\cdot, \cdot)$ and $\mathcal{B}_u(\cdot, \cdot)$ are continuous, i.e.

$$\begin{aligned} |\mathcal{A}_u(y_1, z_1)| &\leqslant C_\mathcal{A} \|y_1\|_V \|z_1\|_V, \qquad y_1, z_1 \in V, \\ |\mathcal{B}_u(y_2, z_2)| &\leqslant C_\mathcal{B} \|y_2\|_W \|z_2\|_W, \qquad y_2, z_2 \in W. \end{aligned} \tag{3}$$

Here $C_\mathcal{A}$ and $C_\mathcal{B}$ are some positive real numbers, not depending on u.

In the sequel, we assume that there exist $c_\mathcal{A}, c_\mathcal{B} > 0$ and $d_\mathcal{B} \geqslant 0$ such that

$$\begin{aligned} \mathcal{A}_u(y, y) + d_\mathcal{B}\, \mathcal{B}_u(y, y) &\geqslant c_\mathcal{A} \|y\|_V^2, \qquad y \in V, \\ \mathcal{B}_u(z, z) &\geqslant c_\mathcal{B} \|z\|_W^2, \qquad z \in W, \end{aligned} \qquad u \in U. \tag{4}$$

Now, for $u \in U$, consider the following eigenvalue problem:

$$\text{find } (\lambda, y) \in \mathbb{R} \times V \setminus \{\vartheta\} : \ \mathcal{A}_u(y, z) = \lambda \mathcal{B}_u(y, z), \quad z \in V. \tag{5}$$

Generalized eigenvalue problems for elliptic boundary-value problems often lead to problems of the form (5) with the properties (1)–(4), and if $d_\mathcal{B} > 0$, then the first inequality in (4) is in effect an abstract Gårding inequality. Notice that a solution (λ, y) of (5) depends on u. In the sequel, in order to emphasize this dependence, we simply write $(\lambda[u], y[u])$.

It is a general fact (see [2]) that under conditions (1)–(4) this problem has a countable set of eigenvalues such that $-d_\mathcal{B} < \lambda_1[u] \leqslant \lambda_2[u] \leqslant \ldots \leqslant \lambda_k[u] \leqslant \ldots$, $\lim_{k \to \infty} \lambda_k[u] = \infty$, each of them being of finite multiplicity. Moreover, a corresponding sequence of eigenfunctions $\{y_k[u]\}_{k \in \mathbb{N}}$ forms a basis in V, this basis being orthogonal with respect to $\mathcal{A}_u(\cdot, \cdot)$. Furthermore, the following useful characterizations for $\lambda_k[u]$ hold (see [10]):

$$\lambda_k[u] = \min_{\substack{Y \leqslant V, \\ \dim Y = k}} \max_{y \in Y \setminus \{\vartheta\}} \frac{\mathcal{A}_u(y, y)}{\mathcal{B}_u(y, y)}, \qquad \lambda_k[u] = \max_{\substack{Y \leqslant V, \\ \operatorname{codim} Y = k-1}} \min_{y \in Y \setminus \{\vartheta\}} \frac{\mathcal{A}_u(y, y)}{\mathcal{B}_u(y, y)}.$$

Notice that, for each $u \in U$, the bilinear form $\mathcal{C}_u(\cdot, \cdot) = \mathcal{A}_u(\cdot, \cdot) + d_{\mathcal{B}} \mathcal{B}_u(\cdot, \cdot)$ is a scalar product on V, which is actually equivalent to $\langle \cdot, \cdot \rangle_s$. In the sequel, in order to emphasize that in a situation V is equipped with $\mathcal{C}_u(\cdot, \cdot)$, we write $V, \mathcal{C}_u(\cdot, \cdot)$. Hereinafter $\perp_u$ denotes the operation of taking orthogonal complements of subsets of V with respect to $\mathcal{C}_u(\cdot, \cdot)$.

Now let $K \subset V$ be a non-trivial closed convex cone with a vertex at ϑ. For $u \in U$, consider the following extremum problem:

$$\text{find } (\mu_1[u], y[u]) \in \mathbb{R} \times K \setminus \{\vartheta\} \ : \ \mu_1[u] = \min_{z \in K \setminus \{\vartheta\}} \frac{\mathcal{A}_u(z, z)}{\mathcal{B}_u(z, z)} = \frac{\mathcal{A}_u(y[u], y[u])}{\mathcal{B}_u(y[u], y[u])}. \tag{6}$$

It is well known (see [11]) that under conditions (1)–(4) there exists a solution to (6). More precisely, the set of elements $y[u]$ minimizing the functional in (6) has the form $\mathcal{K}_u \setminus \{\vartheta\}$, where $\mathcal{K}_u \subset K$ is a closed convex cone with a vertex at ϑ. Furthermore, $\mu_1[u]$ and $y[u]$ are the least positive eigenvalue and its non-trivial solution, respectively, of the following variational inequality:

$$(\mu, y) \in \mathbb{R} \times K \setminus \{\vartheta\} \ : \ \mathcal{A}_u(y, z - y) \geqslant \mu \mathcal{B}_u(y, z - y), \quad \forall z \in K. \tag{7}$$

On the other part, if $\mu_1[u]$ and $y[u]$ are the least positive eigenvalue and its associated eigenfunction of (7), then (6) holds.

Let $\mathcal{U}$ be a non-empty weak* compact subset of U. In this paper, we are primarily interested in the following eigenvalue optimization problems:

$$\text{find } \hat{v} \in \mathcal{U} \ : \ \lambda_k[\hat{v}] = \sup_{u \in \mathcal{U}} \lambda_k[u], \tag{8}$$

$$\text{find } \check{v} \in \mathcal{U} \ : \ \lambda_k[\check{v}] = \inf_{u \in \mathcal{U}} \lambda_k[u], \tag{9}$$

$$\text{find } \hat{w} \in \mathcal{U} \ : \ \mu_1[\hat{w}] = \sup_{u \in \mathcal{U}} \mu_1[u], \tag{10}$$

$$\text{find } \check{w} \in \mathcal{U} \ : \ \mu_1[\check{w}] = \inf_{u \in \mathcal{U}} \mu_1[u]. \tag{11}$$

In the following section, we specify conditions imposed on $a_{\alpha\beta}(\cdot, \cdot)$, $b_{\alpha\beta}(\cdot, \cdot)$ under which these problems are solvable.

2.   Main results

Since $\mathcal{U}$ is weak* compact, it is sufficient to establish the semicontinuity of $\lambda_k[\cdot]$ and $\mu_1[\cdot]$ in the weak* topology on U to prove the existence of solutions to problems (8)–(11). Since $L^1(\Omega)$ is separable, the weak* topology on $L^\infty(\Omega)$ is metrizable whenever one is restricted to bounded sets. Taking into account this and the fact that U is bounded in $L_r^\infty(\Omega)$, it suffices to ascertain the sequential weak* semicontinuity of the functionals under study on U.

The following conventions will be useful in the sequel. Define $\mathcal{Q}_+$ to be the set of all maps $(x, \xi) \mapsto f(x, \xi) : \Omega \times \mathcal{G} \to \mathbb{R}$ such that the following conditions hold true:

i) $(x, \xi) \mapsto f(x, \xi)$ is bounded for a.e. $(x, \xi) \in \Omega \times \mathcal{G}$,

ii) f satisfies the Carathéodory conditions, and

iii) $f(x, \cdot) : \mathcal{G} \to \mathbb{R}$ is convex for a.e. $x \in \Omega$.

Next, define $\mathcal{Q}_- = \{f : -f \in \mathcal{Q}_+\}$, $\mathcal{Q}_0 = \mathcal{Q}_+ \cap \mathcal{Q}_-$. Also, an element $f \in \mathcal{Q}_0$ is said to be in $\mathcal{Q}_c \subset \mathcal{Q}_0$ if and only if $f(x, \cdot) : \mathcal{G} \to \mathbb{R}$ is constant for a.e. $x \in \Omega$. Let θ denote a map from $\Omega \times \mathcal{G}$

to $\mathbb{R}$ such that $\theta(x, \xi) = 0$ for a.e. $(x, \xi) \in \Omega \times \mathcal{G}$. Finally, we say that $f \in \mathcal{Q}_+^* \subset \mathcal{Q}_+$ if and only if the map

$$(x, \xi, \eta) \mapsto f(x, \xi) \eta^2 \ : \ \Omega \times \mathcal{G} \times \mathbb{R} \to \mathbb{R} \tag{12}$$

is convex in (ξ, η) for almost every $x \in \Omega$.

Before proving the main results we need a preliminary lemma.

Lemma 2.1. *Let $\mathcal{F} : L_r^\infty(\Omega) \times L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ be the map given by*

$$\mathcal{F}(u, y, z) = \int_\Omega f\left(x, u(x)\right) y(x)\, z(x)\, dx.$$

Then the following implications hold:

(a) $\left\{ \begin{array}{l} f \in \mathcal{Q}_0, \quad u^n \overset{*}{\rightharpoonup} u \ \ in \ \ L_r^\infty(\Omega), \\ y^n \to y, \ z^n \to z \ \ in \ \ L^2(\Omega) \end{array} \right\} \quad \Rightarrow \quad \lim_{n \to \infty} \mathcal{F}(u^n, y^n, z^n) = \mathcal{F}(u, y, z),$

(b) $\left\{ \begin{array}{l} f \in \mathcal{Q}_c, \quad u^n \overset{*}{\rightharpoonup} u \ \ in \ \ L_r^\infty(\Omega), \\ y^n \to y, \ z^n \rightharpoonup z \ \ in \ \ L^2(\Omega) \end{array} \right\} \quad \Rightarrow \quad \lim_{n \to \infty} \mathcal{F}(u^n, y^n, z^n) = \mathcal{F}(u, y, z),$

(c) $f \in \mathcal{Q}_+, \quad u^n \overset{*}{\rightharpoonup} u \ \ in \ \ L_r^\infty(\Omega), \quad y^n \to y \ \ in \ \ L^2(\Omega) \Rightarrow \liminf\limits_{n \to \infty} \mathcal{F}(u^n, y^n, y^n) \geqslant \mathcal{F}(u, y, y),$

(d) $f \in \mathcal{Q}_+^*, \quad u^n \overset{*}{\rightharpoonup} u \ \ in \ \ L_r^\infty(\Omega), \quad y^n \rightharpoonup y \ \ in \ \ L^2(\Omega) \Rightarrow \liminf\limits_{n \to \infty} \mathcal{F}(u^n, y^n, y^n) \geqslant \mathcal{F}(u, y, y).$

Proof. Parts (a) and (b) are trivial. The proof of (c) is rather similar to the proof of the well-known Tonelli's theorem [12].

Let us prove (d). Firstly, notice that a map of the form (12), where $f \in \mathcal{Q}_+^*$, also satisfies the Carathéodory conditions in the sense that this map is continuous in (ξ, η) for a.e. $x \in \Omega$ and is measurable in x for each $(\xi, \eta) \in \mathcal{G} \times \mathbb{R}$. Secondly, it is easy to show that $f(x, \xi) \eta^2 \geqslant 0$ for almost all $(x, \xi, \eta) \in \Omega \times \mathcal{G} \times \mathbb{R}$. Finally, as weak* convergence in $L^\infty(\Omega)$ implies weak convergence in $L^2(\Omega)$, the part (d) directly follows from [13, Theorem 7]. $\qquad \square$

Recall that $a_{\alpha\beta} = a_{\beta\alpha}$ and $b_{\alpha\beta} = b_{\beta\alpha}$. We use this fact to simplify the formulations of the theorems of this section.

Theorem 2.1. *Let*

$$\begin{array}{llll} a_{\alpha\alpha} \in \mathcal{Q}_-, & |\alpha| \leqslant s, & a_{\alpha\beta} \in \mathcal{Q}_0, & \alpha \neq \beta, \quad |\alpha|, |\beta| \leqslant s, \\ b_{\alpha\alpha} \in \mathcal{Q}_+, & |\alpha| \leqslant m, & b_{\alpha\beta} \in \mathcal{Q}_0, & \alpha \neq \beta, \quad |\alpha|, |\beta| \leqslant m. \end{array}$$

Then $u \mapsto \lambda_k[u]$, $u \mapsto \mu_1[u]$ are weak* upper-semicontinuous over U.

Proof. Let us prove the weak* upper-semicontinuity of $u \mapsto \lambda_k[u]$ over U. Suppose for the sake of contradiction that this is not true. Then for some $k \in \mathbb{N}$ there exists a sequence $\{u^n\}$ such that $u^n \overset{*}{\rightharpoonup} u$ in U, but $\limsup_{n \to \infty} \lambda_k[u^n] > \lambda_k[u]$. Without any loss of generality it can be assumed that

$$\lim_{n \to \infty} \lambda_k[u^n] > \lambda_k[u]. \tag{13}$$

Let $Y_k = \mathrm{span}\,\{y_1[u], \dots, y_k[u]\}$. Define a function $z : \mathbb{R}^k \to Y_k$ by the formula $z(c) = \sum_{i=1}^k c_i y_i[u]$. Further, for each $n \in \mathbb{N}$, let $\chi_n : \mathbb{R}^k \setminus \{0\} \to \mathbb{R}$ be a map given by

$$\chi_n(c) = \frac{\mathcal{A}_{u^n}(z(c), z(c))}{\mathcal{B}_{u^n}(z(c), z(c))}.$$

It is easy to show that $\chi_n(\cdot)$ attains its maximum at some point c^n of the unit sphere S^{k-1} and

$$\chi_n(c^n) \geqslant \lambda_k[u^n]. \tag{14}$$

Without loss of generality assume that $c^n \to c^* \in S^{k-1}$. Let $z^n = z(c^n)$ and $z^* = z(c^*)$. Clearly,

$$z^n \to z^* \ \text{in} \ V, \qquad z^* \in Y_k \setminus \{\vartheta\}. \tag{15}$$

Taking into account the assumptions of the theorem and (15), we can use Lemma 2.1 to deduce

$$\limsup_{n \to \infty} \mathcal{A}_{u^n}(z^n, z^n) \leqslant \mathcal{A}_u(z^*, z^*), \qquad \liminf_{n \to \infty} \mathcal{B}_{u^n}(z^n, z^n) \geqslant \mathcal{B}_u(z^*, z^*). \tag{16}$$

Since $z^* \neq \vartheta$, we can choose a positive number ε such that $\varepsilon < \mathcal{B}_u(z^*, z^*)$. Then for sufficiently large $N(\varepsilon)$, we get that

$$\begin{aligned}
\mathcal{A}_{u^n}(z^n, z^n) &< \limsup_{n \to \infty} \mathcal{A}_{u^n}(z^n, z^n) + \varepsilon, \qquad \lambda_k[u^n] > \lim_{n \to \infty} \lambda_k[u^n] - \varepsilon, \\
\mathcal{B}_{u^n}(z^n, z^n) &> \liminf_{n \to \infty} \mathcal{B}_{u^n}(z^n, z^n) - \varepsilon, \qquad n \geqslant N(\varepsilon).
\end{aligned} \tag{17}$$

In view of (14), (16) and (17), we have that

$$\lim_{n \to \infty} \lambda_k[u^n] - \varepsilon < \lambda_k[u^n] < \frac{\mathcal{A}_u(z^*, z^*) + \varepsilon}{\mathcal{B}_u(z^*, z^*) - \varepsilon}, \qquad n \geqslant N(\varepsilon).$$

Since ε can be chosen arbitrarily small and $z^* \in Y_k \setminus \{\vartheta\}$, we finally obtain that

$$\lim_{n \to \infty} \lambda_k[u^n] \leqslant \frac{\mathcal{A}_u(z^*, z^*)}{\mathcal{B}_u(z^*, z^*)} \leqslant \lambda_k[u],$$

contradicting (13). Hence, $u \mapsto \lambda_k[u]$ is a weak* upper-semicontinuous functional over U. The preceding arguments can be easily applied to prove the weak* upper-semicontinuity of $u \mapsto \mu_1[u]$ over U. $\qquad \square$

Let us now turn to establishing the following weak* lower-semicontinuity criterion.

Theorem 2.2. *Let*

$$\begin{aligned}
a_{\alpha\alpha} &\in \mathcal{Q}_+, & |\alpha| &\leqslant l, & a_{\alpha\beta} &\in \mathcal{Q}_0, & \alpha &\neq \beta, & |\alpha|, |\beta| &\leqslant l, \\
a_{\alpha\alpha} &\in \mathcal{Q}_+^*, & l < |\alpha| &\leqslant s, & a_{\alpha\beta} &\in \mathcal{Q}_c, & |\alpha| &\leqslant l < |\beta| \leqslant s, \\
& & & & a_{\alpha\beta} &= \theta, & \alpha &\neq \beta, & l < |\alpha|, |\beta| &\leqslant s, \\
b_{\alpha\alpha} &\in \mathcal{Q}_-, & |\alpha| &\leqslant m, & b_{\alpha\beta} &\in \mathcal{Q}_0, & \alpha &\neq \beta, & |\alpha|, |\beta| &\leqslant m.
\end{aligned}$$

Then $u \mapsto \lambda_k[u]$, $u \mapsto \mu_1[u]$ are weak lower-semicontinuous over U.*

Proof. Let us prove the weak* lower-semicontinuity of $\lambda_k[\cdot]$. From the minimax principle, (3), (4) it follows that there exist two sequences $\{\check{\lambda}_k\}_{k \in \mathbb{N}}$, $\{\hat{\lambda}_k\}_{k \in \mathbb{N}}$ such that $\check{\lambda}_k \leqslant \lambda_k[v] \leqslant \hat{\lambda}_k, \forall v \in U$. Let k be a natural number, $u \in U$, and $\pi_k : U \to \mathbb{R}$ be a map defined as

$$\pi_k(v) = \min_{y \in Z_k \setminus \{\vartheta\}} \frac{\mathcal{A}_v(y, y)}{\mathcal{B}_v(y, y)}, \qquad Z_k = \operatorname{span} \{\vartheta, y_1[u], \ldots, y_{k-1}[u]\}^{\perp_u}.$$

Clearly, $Z_k \leqslant V$, codim $Z_k = k - 1$. Assume that there exists a sequence $\{u^n\}_{n \in \mathbb{N}}$ such that

$$u^n \overset{*}{\rightharpoonup} u \ \text{in} \ U \quad \Rightarrow \quad \lim_{n \to \infty} \lambda_k[u^n] < \lambda_k[u]. \tag{18}$$

Let z^n be an element in Z_k such that

$$d_{\mathcal{B}} + \pi_k(u^n) = [\mathcal{B}_{u^n}(z^n, z^n)]^{-1}, \qquad \mathcal{C}_{u^n}(z^n, z^n) = 1.$$

Clearly, $\|z^n\|_V^2 \leqslant c_{\mathcal{A}}^{-1}$, $\mathcal{C}_u(z^n, z^n) \leqslant c_{\mathcal{A}}^{-1}(C_{\mathcal{A}} + d_{\mathcal{B}} C_{\mathcal{B}})$. The imbedding of V in $\mathcal{W}$ being compact, it can be assumed that

$$z^n \to z^* \text{ in } \mathcal{W}, \qquad \partial^\alpha z^n \rightharpoonup \partial^\alpha z^* \text{ in } L^2(\Omega), \quad l < |\alpha| \leqslant s, \qquad z^n \rightharpoonup z^* \text{ in } V, \, \mathcal{C}_u(\cdot, \cdot). \quad (19)$$

Taking into consideration the conditions of the theorem and (19), we can apply Lemma 2.1 to obtain that

$$\liminf_{n \to \infty} \mathcal{A}_{u^n}(z^n, z^n) \geqslant \mathcal{A}_u(z^*, z^*), \qquad \limsup_{n \to \infty} \mathcal{B}_{u^n}(z^n, z^n) \leqslant \mathcal{B}_u(z^*, z^*). \quad (20)$$

Since

$$\|z^n\|_W^2 \geqslant \left[C_{\mathcal{B}} \left(\hat{\lambda}_k + d_{\mathcal{B}} \right) \right]^{-1}, \qquad \mathcal{C}_u(z^n, y) \to \mathcal{C}_u(z^*, y), \quad y \in V,$$

we obtain that $z^* \in Z_k \setminus \{\vartheta\}$.

Now let $\varepsilon > 0$ be given, and choose $N(\varepsilon)$ sufficiently large such that

$$\mathcal{A}_{u^n}(z^n, z^n) > \liminf_{n \to \infty} \mathcal{A}_{u^n}(z^n, z^n) - \varepsilon, \qquad \limsup_{n \to \infty} \pi_k(u^n) > \pi_k(u^n) - \varepsilon,$$

$$\mathcal{B}_{u^n}(z^n, z^n) < \limsup_{n \to \infty} \mathcal{B}_{u^n}(z^n, z^n) + \varepsilon, \qquad n \geqslant N(\varepsilon). \quad (21)$$

By virtue of (20) and (21), we get that

$$\frac{\mathcal{A}_u(z^*, z^*) - \varepsilon}{\mathcal{B}_u(z^*, z^*) + \varepsilon} < \limsup_{n \to \infty} \pi_k(u^n) + \varepsilon.$$

Since ε was arbitrary, and $z^* \in Z_k \setminus \{\vartheta\}$, it immediately follows that

$$\lambda_k[u] = \pi_k(u) \leqslant \frac{\mathcal{A}_u(z^*, z^*)}{\mathcal{B}_u(z^*, z^*)} \leqslant \limsup_{n \to \infty} \pi_k(u^n) \leqslant \lim_{n \to \infty} \lambda_k[u^n].$$

The preceding inequalities contradict (18). Consequently, $u \mapsto \lambda_k[u]$ is a weak* lower-semicontinuous functional over U. The proof of the weak* lower-semicontinuity of $\mu_1[\cdot]$ is omitted due to its similarity to the foregoing proof. $\qquad \square$

We can combine the preceding results to obtain a criterion for the weak* continuity of $u \mapsto \mu_1[u]$ over U. However it is reasonable to apply another approach to get a stronger result.

Theorem 2.3. *Let $a_{\alpha\beta} \in \mathcal{Q}_c$, $l < |\alpha|$, $a_{\alpha\beta} \in \mathcal{Q}_0$, $|\alpha|, |\beta| \leqslant l$, $b_{\alpha\beta} \in \mathcal{Q}_0$, $|\alpha|, |\beta| \leqslant m$. Then $u \mapsto \mu_1[u]$ is weak* continuous on U.*

Let $u^n \xrightarrow{*} u$ in U. Using similar arguments as in [6, Prop. 4.3], we obtain that $\mu_1[u] \leqslant \liminf_{n \to \infty} \mu_1[u^n]$. Applying Theorem 2.1, we get that $\mu_1[u] = \lim_{n \to \infty} \mu_1[u^n]$.

Though this sketch can be directly applied to the proof of the weak* continuity of $\lambda_k[\cdot]$ provided that the hypotheses of Theorem 2.3 hold, we remark that it is not difficult to obtain this result from continuity of a finite system of eigenvalues with respect to generalized convergence of closed operators [14].

Since each lower (upper) semicontinuous functional attains its minimum (maximum) over a non-empty compact set, we directly obtain existence criteria for (8)–(11) from the previous

theorems. Note that we can also apply the obtained results to get existence theorems for another extremum problems. Let us give an example. Assume that U is weak* compact. Let

$$\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R} \ : \ \mathcal{V}_n = \left\{ u \in U \ : \ \lambda_i[u] \geqslant \tilde{\lambda}_i, \ i = 1, \ldots, n \right\} \neq \varnothing. \tag{22}$$

Suppose that $u \mapsto \lambda_i[u]$ is weak* upper-semicontinuous on U for each $i \in \{1, \ldots, n\}$. Then $\mathcal{V}_n$ is weak* compact. Now consider the following problem:

$$\min_{u \in \mathcal{V}_n} \int_\Omega f(x, u(x)) \, dx, \quad f \in \mathcal{Q}_+.$$

Using Lemma 2.1, we get that this problem is solvable.

In the following section, we give some concrete applications of the obtained results.

3.   Some applications

Throughout this section, all quantities are considered as dimensionless, and we consider that $H_0^2(\Omega) \subset V \leqslant H^2(\Omega)$, $W = \mathcal{W} = H^1(\Omega)$.

Optimal design of columns subjected to buckling. Let $\Omega = (0, 1)$, $e, u, \kappa, \rho \in L^\infty(\Omega)$, and $e(x) \geqslant e_0 > 0$, $u(x) \geqslant u_0 > 0$, $\kappa(x) \geqslant 0$, $\rho(x) \geqslant \rho_0 > 0$ a.e. in Ω. Define $\mathcal{A}_u : V \times V \to \mathbb{R}$ and $\mathcal{B}_u : V \times V \to \mathbb{R}$ by

$$\mathcal{A}_u(y, z) = \int_0^1 \left(e u^\nu y'' z'' + \kappa y z \right) dx, \quad \mathcal{B}_u(y, z) = \int_0^1 y' z' \, dx. \tag{23}$$

The variation equation describing buckling of a non-homogeneous column lying on an elastic foundation is (5), where $\mathcal{A}_u(\cdot, \cdot)$, $\mathcal{B}_u(\cdot, \cdot)$ are expressed by (23). Here e, ρ are Young's modulus and the density of the column material, respectively, κ is the foundation modulus, u corresponds to the cross-sectional area distribution of the column, the lowest eigenvalue of the boundary-value problem defined by (5), (23) is usually connected with the critical load that causes buckling of the column, ν is a positive parameter. For example, the case $\nu = 1$ corresponds to thin-walled columns, u being an affine function of the cross-sectional area distribution (see [7] for more details). Here we consider the following boundary conditions: (i) *clamped-clamped* ($V = H_0^2(\Omega)$); (ii) *simply supported* ($V = \{v \in H^2(\Omega) : v(0) = v(1) = 0\}$); (iii) *clamped-simply supported*. Now let

$$0 < \check{u}, \hat{u}, \tilde{m} < +\infty \ : \ U = \{u \in L^\infty(\Omega) \ : \ \check{u} \leqslant u(x) \leqslant \hat{u} \text{ a.e. in } \Omega\} \neq \varnothing,$$

$$\mathcal{U} = \left\{ u \in U \ : \ \int_\Omega \rho(x) u(x) \, dx \leqslant \tilde{m} \right\} \neq \varnothing. \tag{24}$$

Clearly, all the assumptions of Section 1 hold. Then we can apply the obtained results to deduce that, for each $k \in \mathbb{N}$, the problem (8) defined by $0 < \nu \leqslant 1$, (5), (23), (24) is solvable. In [7] S. Cox and M. Overton established existence for the problem (8) defined by (5), (23), $k = 1$, $e = 1$, $\kappa = 0$, $\nu > 0$, $\mathcal{U} = \{u \in U : \|u\|_1 = 1\}$. Notice that the case $\nu > 1$ is not covered by the theorems of this paper. However, their proof is essentially based on symmetry considerations, the one-dimensionality of the problem, the properties of $\mathcal{U}$, and the fact that there exists a non-negative eigenfunction associated with the lowest eigenvalue. Meanwhile, there is no need in any information concerning eigenfunctions to apply the results of Section 2, $\mathcal{U}$ can be any weak* compact set of functions having a common positive lower bound, and the functions e, κ, ρ can be non-symmetric about $x = \dfrac{1}{2}$.

Now let us give an example for the case of variational inequalities. Assume that the column is clamped on one end and unilaterally supported on another one. In this case, the set of admissible states is

$$K = \{z \in H^2(\Omega) : z(0) = z'(0) = 0, z(1) \geqslant 0\}. \qquad (25)$$

Clearly, K is a closed convex cone with a vertex at ϑ. The variational inequality describing buckling of the column on the elastic foundation is (7), $\mathcal{A}_u(\cdot,\cdot)$, $\mathcal{B}_u(\cdot,\cdot)$ and K being defined by (23), (25). As above, we conclude that the problem (10) determined by $0 < \nu \leqslant 1$, (7), (23)–(25) is solvable.

Finally, let us consider the following problem of minimization of the mass of a column with a lower limit on the critical load:

$$m[u] = \int_\Omega \rho(x)u(x)\,dx \to \min, \quad u \in \mathcal{V}_1. \qquad (26)$$

Here $\mathcal{V}_1$ is given by (22). If $0 < \nu \leqslant 1$, then $u \mapsto \lambda_1[u]$ is weak* upper-semicontinuous over U. In turn, $u \mapsto m[u]$ is weak* continuous over $\mathcal{V}_1$. From the above it follows that the problem (26) defined by (5), (22)–(24) is solvable. It can be verified easily that an analogous result holds for the case of variational inequalities.

Optimal design of a vibrating three-layered plate. Since regularization techniques used in [9] do not ensure existence of optimal solutions to the original problem, it is interesting to give a model for plates for which the results of Section 2 are applicable. Let us consider a model for three-layered plates ignoring shears in the middle layer. For a more comprehensive treatment of the model, the reader is referred to [2, 4.3]. Let Ω be a bounded domain of $\mathbb{R}^2$ having the cone property, and U be determined by (24). Define $\mathcal{A}_u : V \times V \to \mathbb{R}$ and $\mathcal{B}_u : V \times V \to \mathbb{R}$ as

$$\mathcal{A}_u(y,z) = \int_\Omega \left[A_1(u)\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 z}{\partial x_1^2} + A_2(u)\frac{\partial^2 y}{\partial x_2^2}\frac{\partial^2 z}{\partial x_2^2} + \right.$$
$$\left. + A_{12}(u)\left(\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 z}{\partial x_2^2} + \frac{\partial^2 y}{\partial x_2^2}\frac{\partial^2 z}{\partial x_1^2}\right) + 2A_3(u)\frac{\partial^2 y}{\partial x_1\partial x_2}\frac{\partial^2 z}{\partial x_1\partial x_2} \right] dx, \qquad (27)$$

$$\mathcal{B}_u(y,z) = \int_\Omega \left[B_1(u)yz + B_2(u)\left(\frac{\partial y}{\partial x_1}\frac{\partial z}{\partial x_1} + \frac{\partial y}{\partial x_2}\frac{\partial z}{\partial x_2}\right) \right] dx,$$

where

$$A_i(u) = \frac{E_{ii}}{2}(u+\tilde{u})^2 u, \quad i = 1,2, \quad A_{12}(u) = \frac{E_{12}}{2}(u+\tilde{u})^2 u, \quad A_3(u) = G(u+\tilde{u})^2 u,$$

$$B_1(u) = 2u\rho + \tilde{u}\tilde{\rho}, \quad B_2(u) = \rho\left(\tilde{u}u^2 + \frac{1}{2}\tilde{u}^2 u + \frac{2}{3}u^3\right) + \frac{1}{12}\tilde{\rho}\tilde{u}^3, \quad E_{ij}, G, \rho, \tilde{\rho} > 0, \qquad (28)$$

$$u \in U, \quad u(x) + \tilde{u}(x) = c \equiv const, \quad \tilde{u}(x) > 0, \quad x \in \Omega.$$

Here E_{ij}, G are the elasticity characteristics of the exterior layers, u is the thickness of the exterior layers, $\tilde{u}$ is the thickness of the middle layer, $\tilde{\rho}$ and ρ are the densities of the material of the interior and extrerior layers. Then the equation describing free oscillations of the plate is (5), where $\mathcal{A}_u(\cdot,\cdot)$, $\mathcal{B}_u(\cdot,\cdot)$ are expressed by (27), (28). Let $\partial\Omega$ denote the boundary of Ω, and let Γ_1, Γ_2 be non-empty parts of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \varnothing$. Assume that Γ_1 contains three points which do not belong to a straight line. For this plate consider the following standard boundary conditions: i) *clamped on* Γ_1 $\left(V = \left\{u \in H^2(\Omega) : u|_{\Gamma_1} = 0, \frac{\partial u}{\partial\nu}|_{\Gamma_1} = 0\right\}\right)$; ii) *simply supported on* Γ_1 $\left(V = \left\{u \in H^2(\Omega) : u|_{\Gamma_1} = 0\right\}\right)$; iii) *clamped on* Γ_1 *and simply supported on* Γ_2. For another interesting implementations of V, the reader is referred to [2, 4.1.4]. It is not difficult

to establish that there exist $c_{\mathcal{A}} > 0$, $d_{\mathcal{B}} \geqslant 0$ such that the first inequality in (4) holds, and all the assumptions of Section 1 are fulfilled. Thus, one can apply the results of Section 2 to obtain that, for each $k \in \mathbb{N}$, the problem (8) defined by (5), (24), (27), (28) is solvable if $2\rho \geqslant \tilde{\rho}$. Since the material of the middle layer is usually comparatively light, we can also consider the problem

$$m[u] = \rho \int_{\Omega} u(x)\, dx \to \min, \quad u \in \mathcal{V}_n,$$

determined by (5), (22), (24), (27), (28), $2\rho \geqslant \tilde{\rho}$, which is solvable. From the above considerations we conclude that for plates clamped on Γ_1 and unilaterally supported on Γ_2 $\left(K = \left\{ z \in H^2(\Omega) : z\big|_{\Gamma_1} = \dfrac{\partial z}{\partial \nu}\big|_{\Gamma_1} = 0,\ z\big|_{\Gamma_2} \geqslant 0 \right\} \right)$ extremum problems analogous to the above ones are solvable as well.

Clearly, the results of Section 2 can be directly applied to other problems in optimal design.

Further investigations in this direction are actively being carried on.

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Критерии существования в некоторых задачах оптимизации, связанных с собственными значениями эллиптических операторов

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Доказываются критерии существования для некоторого класса задач оптимизации, связанных с собственными значениями линейных эллиптических краевых задач (в том числе в форме вариационных неравенств). Применяемый метод позволяет сформулировать аналогичные критерии для экстремальных задач, связанных с собственными значениями нелинейных краевых задач. Приводятся приложения к оптимальному проектированию конструкций, дается сравнение полученных результатов с известными.

Ключевые слова: задача оптимизации собственного значения, эллиптическая краевая задача, вариационное неравенство, теорема существования, оптимальное проектирование конструкций.