УДК 517.532

Axisymmetric Thermocapillary Motion in a Cylinder at Small Marangoni Number

Evgeniy P. Magdenko*
Institute of Computational Modeling SB RAS
Akademgorodok, 50/44, Krasnoyarsk, 660036
Russia

Received 21.04.2015, received in revised form 05.05.2015, accepted 20.06.2015

The solution to the linear problem of axisymmetric thermocapillary motion of two non-miscible viscous fluids in a cylindrical tube is presented. Their common interface is fixed and undeformable. This problem is an inverse problem because pressure gradients are unknown functions. The solution of the non-stationary problem is presented in the form of analytical expressions. They are obtained with the use of the method of Laplace transformation. If the wall temperature is stabilized then the general solution tends to the stationary solution as time increases. Numerical calculations confirm the theoretical results.

Keywords: thermocapillary motion, inverse problem, Laplace transform, fluid-fluid interface.
DOI: 10.17516/1997-1397-2015-8-3-303-311

1. Problem statement

The axisymmetric motion of viscous thermally conducting fluid in a cylindrical system of coordinates is described by the Navier-Stocks equations

$$u_{1t} + u_1 u_{1r} + v_1 u_{1z} + \frac{1}{\rho} p_r = \nu \left(\Delta u_1 - \frac{u_1}{r^2} \right), \tag{1.1}$$

$$v_{1t} + u_1 v_{1r} + v_1 v_{1z} + \frac{1}{\rho} p_z = \nu \Delta v_1, \tag{1.2}$$

$$u_{1r} + \frac{1}{r} u_1 + v_{1z} = 0, \tag{1.3}$$

$$\theta_t + u_1 \theta_r + v_1 \theta_z = \chi \Delta \theta, \tag{1.4}$$

where $u_1(r,z,t), v_1(r,z,t)$ are the projections the velocity vector on the axes r, z; $p(r,z,t)$ is the pressure; $\theta(r,z,t)$ is the deviation of the temperature from the equilibrium value; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the Laplace operator, ρ, ν, χ are density, kinematic viscosity and thermal diffusivity, respectively.

System of equtions (1.1)–(1.4) admits of subgroup of four-dimensional continuous transformations [1]. They are generated by the operators $\langle \partial_z,\, t\partial_z + \partial_{v_1},\, \partial_p,\, \partial_\theta \rangle$. Their invariants are t, r, u. Therefore, partially invariant solutions of rank 2 and 3 should be sought in the form [2]

$$u_1 = u_1(r,t), \quad v_1 = v_1(r,z,t), \quad p = p(r,z,t), \quad \theta = \theta(r,z,t). \tag{1.5}$$

*magdenko_evgeniy@icm.krasn.ru

In this case, it follows from the equation of conservation of mass (1.3) that v_1 is a linear function of z:

$$v_1 = w(r,t)z + w_1(r,t). \tag{1.6}$$

Moreover, we have

$$rw + (u_1 r)_r = 0. \tag{1.7}$$

The momentum equations (1.1), (1.2) give us the following relations

$$w_t + u_1 w_r + w^2 = \nu \left(w_{rr} + \frac{1}{r} w_r \right) + h(t), \tag{1.8}$$

$$\frac{1}{\rho} p = l(t) - \nu w_r - \frac{u_1^2}{2} - \frac{\partial}{\partial t} \int u_1(r,t)\, dr - \frac{h(t)}{2} z^2 \tag{1.9}$$

with arbitrary functions $h(t)$ и $l(t)$.

We use solutions (1.5)–(1.9) to describe the motion in a cylindrical tube of radius b with the fluid-fluid interface at $a < b$. We assume for simplicity that $w_1 = 0$ in (1.6). If to write down the problem in dimensionless form, then the nonlinear term will stand Marangoni number $\mathrm{M} = æ\theta a^2/\rho\nu\chi$. It is assumed that $\mathrm{M} \ll 1$, that is last performed in thin layers or a very high viscosities. As a result, we obtain the following problem

$$w_{1t} = \nu_1 \left(w_{1rr} + \frac{1}{r} w_{1r} \right) + h_1(t), \quad 0 < r < a, \tag{1.10}$$

$$w_{2t} = \nu_2 \left(w_{2rr} + \frac{1}{r} w_{2r} \right) + h_2(t), \quad a < r < b, \tag{1.11}$$

$$w_1(r,0) = w_{10}(r), \quad w_2(r,0) = w_{20}(r), \tag{1.12}$$

$$w_1(a,t) = w_2(a,t), \quad (\mu_2 w_{2r} - \mu_1 w_{1r})z = -\frac{\partial \sigma}{\partial z}, \tag{1.13}$$

$$\theta_1(a,z,t) = \theta_2(a,z,t), \quad \left(k_1 \frac{\partial \theta_1}{\partial r} - k_2 \frac{\partial \theta_2}{\partial r} \right) \Big|_{r=a} = 0. \tag{1.14}$$

Assume that the surface tension linearly depends on temperature:

$$\sigma = \sigma^0 - æ(\theta - \theta_0), \quad æ = \mathrm{const} > 0. \tag{1.15}$$

It is obvious that for the temperature we have following representations

$$\theta_j(r,z,t) = a_j(r,t)z^2 + b_j(r,t), \quad j = 1,2. \tag{1.16}$$

Taking into account (1.15), the second boundary condition (1.13) can be rewritten as

$$\mu_2 w_{2r}(a,t) - \mu_1 w_{1r}(a,t) = 2æ a_1(a,t). \tag{1.17}$$

Functions $a_j(r,t)$, $b_j(r,t)$ satisfy the following equations

$$a_{1t} = \chi_1 \left(a_{1rr} + \frac{1}{r} a_{1r} \right), \quad 0 < r < a, \tag{1.18}$$

$$a_{2t} = \chi_2 \left(a_{2rr} + \frac{1}{r} a_{2r} \right), \quad a < r < b, \tag{1.19}$$

$$b_{1t} = \chi_1 \left(b_{1rr} + \frac{1}{r} b_{1r} \right) + 2\chi_1 a_1, \quad 0 < r < a, \tag{1.20}$$

$$b_{2t} = \chi_1 \left(b_{2rr} + \frac{1}{r} b_{2r} \right) + 2\chi_2 a_2, \quad a < r < b, \tag{1.21}$$

$$a_j(r,0) = a_{j0}(r), \quad b_j(r,0) = b_{j0}(r). \tag{1.22}$$

It is necessary to add conditions of the boundedness of $w_1(r,t)$, $a_1(r,t)$, $b_1(r,t)$ at $r = 0$ and no-slip conditions at $r = b$:

$$w_2(b,t) = 0, \quad \int_0^a r w_1(r,t)\, dr = 0, \quad \int_a^b r w_2(r,t)\, dr = 0. \tag{1.23}$$

The last two relations follow from equation (1.7). They allow us define the functions $h_1(t)$ and $h_2(t)$ if function $a_1(r,t)$ is known. So at first we find functions $a_j(r,t)$. Taking into account (1.16) and boundary conditions (1.14), we write

$$a_1(a,t) = a_2(a,t), \quad k_1 \frac{\partial a_1(a,t)}{\partial r} = k_2 \frac{\partial a_2(a,t)}{\partial r}. \tag{1.24}$$

Functions $b_1(r,t)$ and $b_2(r,t)$ admit similar conditions. In addition, on the solid wall $r = b$ the temperature is given: $\theta_2(b,z,t) = \alpha(t)z^2 + \beta(t)$, where functions $\alpha(t)$ and $\beta(t)$ are known. This means that

$$a_2(b,t) = \alpha(t), \quad b_2(b,t) = \beta(t). \tag{1.25}$$

2. The stationary solution

Let us assume that all functions do not depend on time. Then from (1.18)–(1.20), (1.24), (1.25) we find that $a_1^s = a_2^s = \text{const} = \alpha^s$ and

$$w_1^s(r) = \frac{a æ \alpha^s}{\mu_2} f_1(\delta) \left(\frac{1}{2} - \frac{r^2}{a^2} \right), \quad \delta = \frac{a^2}{b^2} < 1, \tag{2.1}$$

$$w_2^s(r) = \frac{2a æ \alpha^s}{\mu_2} \frac{1}{f_3(\delta)} \left[\frac{(1-\delta)^2}{1 - \delta + \delta \ln \delta} \ln \left(\frac{r}{b} \right) + 1 - \frac{r^2}{b^2} \right], \tag{2.2}$$

where

$$f_1(\delta) = \frac{2\delta f_2(\delta)}{f_3(\delta)}, \quad f_2(\delta) = \frac{\delta - 1}{\delta} \left[2 + \frac{(1-\delta)\ln\delta}{1 - \delta + \delta \ln \delta} \right], \tag{2.3}$$

$$f_3(\delta) = \frac{(1-\delta)^2}{1 - \delta + \delta \ln \delta} - 2\delta + 2\mu\delta f_2(\delta), \quad \mu = \frac{\mu_1}{\mu_2}, \tag{2.4}$$

$$h_1^s = \frac{4æ\nu\delta f_1(\delta)\alpha^s}{a\rho_2 f_3(\delta)}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad h_2^s = \frac{8æ\delta\alpha^s}{a\rho_2 f_3(\delta)}. \tag{2.5}$$

The dependence of velocity components on r is given by formulas

$$u_1^s = -\frac{a^2 æ \alpha^s}{4\mu_2} \left(\frac{r}{a} \right) \left(1 - \frac{r^2}{a^2} \right), \tag{2.6}$$

$$u_2^s = -\frac{a^2 æ \alpha^s}{\mu_2 f_3(\delta)} \left(\frac{a}{r} \right) \left\{ \frac{(1-\delta)^2}{2(1 - \delta + \delta \ln \delta)} \left[\frac{r^2}{a^2} \ln \left(\frac{\delta r^2}{a^2} \right) - \right. \right.$$
$$\left. \left. - \ln \delta + 1 - \frac{r^2}{a^2} \right] + \frac{r^2}{a^2} - \frac{\delta}{2} \frac{r^4}{a^4} - 1 + \frac{\delta}{2} \right\}. \tag{2.7}$$

Function $b_j^s(r)$ are

$$b_1^s(r) = \frac{\alpha^s b^2}{2}\left[1 + (1-k)\delta\ln\delta - \frac{\delta r^2}{a^2}\right] + \beta^s, \quad 0 \leqslant r \leqslant a, \tag{2.8}$$

$$b_2^s(r) = \frac{\alpha^s b^2}{2}\left[1 - \frac{\delta r^2}{a^2} + (1-k)\delta\ln\left(\frac{\delta r^2}{a^2}\right)\right] + \beta^s, \quad 0 \leqslant a \leqslant b, \tag{2.9}$$

where $k = k_1/k_2$.

3. Solution of the problem by the method of the Laplace transform

To solve linear adjoint problems one can use the Laplace transform [3]. It is defined as follows

$$\tilde{a}_j(r,s) = \int_0^\infty a_j(r,t)e^{-st}\,dt, \quad j = 1,2. \tag{3.1}$$

Then the problem is reduced to a boundary value problem for ordinary differential equation

$$\tilde{a}_{1rr} + \frac{1}{r}\,\tilde{a}_{1r} - \frac{s}{\chi_1}\,\tilde{a}_1 = -a_{10}(r), \quad 0 < r < a, \tag{3.2}$$

$$\tilde{a}_{2rr} + \frac{1}{r}\,\tilde{a}_{2r} - \frac{s}{\chi_2}\,\tilde{a}_2 = -a_{20}(r), \quad a < r < b, \tag{3.3}$$

$$\begin{gathered}\tilde{a}_1(a,s) = \tilde{a}_2(a,s), \quad k_1\frac{\partial\tilde{a}_1(a,s)}{\partial r} = k_2\frac{\partial\tilde{a}_2(a,s)}{\partial r}, \\ \tilde{a}_2(b,s) = \tilde{\alpha}(s).\end{gathered} \tag{3.4}$$

General solution of equations (3.2), (3.3) can be represented in the form (the condition of boundedness of $\tilde{a}_1$ at $r = 0$ is taken into account)

$$\tilde{a}_1(r,s) = C_1 I_0\left(\sqrt{\frac{s}{\chi_1}}\,r\right) + \int_0^r \tau\left[I_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)K_0\left(\sqrt{\frac{s}{\chi_1}}\,r\right) - \right.$$
$$\left. - I_0\left(\sqrt{\frac{s}{\chi_1}}\,r\right)K_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)\right]a_{10}(\tau)\,d\tau, \tag{3.5}$$

$$\tilde{a}_2(r,s) = C_2 I_0\left(\sqrt{\frac{s}{\chi_2}}\,r\right) + C_3 K_0\left(\sqrt{\frac{s}{\chi_2}}\,r\right) + \int_a^r \tau\left[I_0\left(\sqrt{\frac{s}{\chi_2}}\,\tau\right)K_0\left(\sqrt{\frac{s}{\chi_2}}\,r\right) - \right.$$
$$\left. - I_0\left(\sqrt{\frac{s}{\chi_2}}\,r\right)K_0\left(\sqrt{\frac{s}{\chi_2}}\,\tau\right)\right]a_{20}(\tau)\,d\tau, \tag{3.6}$$

where $I_0(x)$, $K_0(x)$ are modified Bessel functions of the 1st and 2nd kind. The quantities C_1, C_2 and C_3 are determined from boundary conditions (3.4)

$$C_1 = \frac{1}{\Delta}\begin{vmatrix} f_1(s) & I_0(z) & K_0(z) \\ f_2(s) & -I_0(y) & -K_0(y) \\ f_3(s) & -\sqrt{\chi}\,I_1(y) & \sqrt{\chi}\,K_1(y) \end{vmatrix},$$

$$C_2 = \frac{1}{\Delta}\begin{vmatrix} 0 & f_1(s) & K_0(z) \\ I_0(x) & f_2(s) & -K_0(y) \\ kI_1(x) & f_3(s) & \sqrt{\chi}\,K_1(y) \end{vmatrix}, \tag{3.7}$$

$$C_3 = \frac{1}{\Delta}\begin{vmatrix} 0 & I_0(z) & f_1(s) \\ I_0(x) & -I_0(y) & f_2(s) \\ kI_1(x) & -\sqrt{\chi}\,I_1(y) & f_3(s) \end{vmatrix},$$

$$\Delta = \begin{vmatrix} 0 & I_0(z) & K_0(z) \\ I_0(x) & -I_0(y) & -K_0(y) \\ kI_1(x) & -\sqrt{\chi}\,I_1(y) & \sqrt{\chi}\,K_1(y) \end{vmatrix},$$

where $x = a\sqrt{s/\chi_1},\ y = a\sqrt{s/\chi_2},\ z = b\sqrt{s/\chi_2},\ \chi = \chi_1/\chi_2,\ k = k_1/k_2,$

$$f_1(s) = \int_a^b \tau\left[I_0(z)K_0\left(\sqrt{\frac{s}{\chi_2}}\,\tau\right) - I_0\left(\sqrt{\frac{s}{\chi_2}}\,\tau\right)K_0(z)\right]a_{20}(\tau)\,d\tau + \tilde{\alpha}(s),$$

$$f_2(s) = -\int_0^a \tau\left[I_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)K_0(x) - I_0(x)K_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)\right]a_{10}(\tau)\,d\tau, \tag{3.8}$$

$$f_3(s) = k\int_0^a \tau\left[I_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)K_1(x) + I_1(x)K_0\left(\sqrt{\frac{s}{\chi_1}}\,\tau\right)\right]a_{10}(\tau)\,d\tau.$$

When $t \to 0$ we have $I_0(t) \sim 1 + t^2/4,\ K_0(t) \sim -\ln(t/2),\ I_1(t) \sim t/2 + t^3/16,\ K_1(t) \sim 1/t + t\ln(t/2)/2$ and

$$\Delta(s) \sim \frac{1}{y}\left\{\frac{kxy}{2}\,\ln\left(\frac{a}{b}\right) - \sqrt{\chi}\left[1 + \frac{y^2}{2}\,\ln\left(\frac{a}{b}\right) + \frac{x^2+z^2}{4}\right]\right\}. \tag{3.9}$$

Therefore from (3.6)–(3.8) we obtain

$$\lim_{s\to 0} s\tilde{a}_j(s) = \lim_{s\to 0} s\tilde{\alpha}(s) = \alpha^s = \text{const}. \tag{3.10}$$

This means that function $a_j(r,t)$ tends to constant value as time increases [3].

Let us turn to the definition of the functions $\tilde{w}_j(r,t)$. The motion arises only under the action of thermocapillary forces, that is, initial conditions (1.12) are zero: $w_{j0}(r) = 0,\ j = 1,2$. Then for the image $\tilde{w}_j(r,s)$ we have the following boundary value problem

$$\tilde{w}_{1rr} + \frac{1}{r}\,\tilde{w}_{1r} - \frac{s}{\nu_1}\,\tilde{w}_1 = -\frac{\tilde{h}_1(s)}{\nu_1}, \quad 0 < r < a, \tag{3.11}$$

$$\tilde{w}_{2rr} + \frac{1}{r}\,\tilde{w}_{2r} - \frac{s}{\nu_2}\,\tilde{w}_2 = -\frac{\tilde{h}_2(s)}{\nu_2}, \quad a < r < b, \tag{3.12}$$

$$\tilde{w}_1(a,s) = \tilde{w}_2(a,s), \tag{3.13}$$

$$\mu_2\tilde{w}_{2r}(a,s) - \mu_1\tilde{w}_{1r}(a,s) = 2æ\tilde{a}_1(a,s), \tag{3.14}$$

$$\tilde{w}_2(b,s)=0, \quad \int_0^a r\tilde{w}_1(r,s)\,dr=0, \quad \int_a^b r\tilde{w}_2(r,s)\,dr=0. \tag{3.15}$$

Here, the function $\tilde{a}_1(a,s)$ is already known from equation (3.5) and $|\tilde{w}_1(0,s)|<\infty$.

The solution of equations (3.11), (3.12) can be represented as

$$\begin{aligned} \tilde{w}_1 &= D_1 I_0\left(\sqrt{\frac{s}{\nu_1}}\,r\right)+\frac{\tilde{h}_1(s)}{s}\,, \\ \tilde{w}_2 &= D_2 I_0\left(\sqrt{\frac{s}{\nu_2}}\,r\right)+D_3 K_0\left(\sqrt{\frac{s}{\nu_2}}\,r\right)+\frac{\tilde{h}_2(s)}{s}\,. \end{aligned} \tag{3.16}$$

The boundary conditions (3.13), (3.14) and the first condition (3.15) allow us to find the values of D_1, D_2 and D_3:

$$D_1=\frac{1}{s\Delta_1}\begin{vmatrix} -\tilde{h}_2 & I_0(z_1) & K_0(z_1) \\ \tilde{h}_2-\tilde{h}_1 & -I_0(y_1) & -K_0(y_1) \\ -\dfrac{2æ\sqrt{\nu_2 s}}{\mu_2}\,\tilde{a}_1(a,s) & -I_1(y_1) & K_1(y_1) \end{vmatrix},$$

$$D_2=\frac{1}{s\Delta_1}\begin{vmatrix} 0 & -\tilde{h}_2 & K_0(z_1) \\ I_0(x_1) & \tilde{h}_2-\tilde{h}_1 & -K_0(y_1) \\ \dfrac{\mu}{\sqrt{\nu}}\,I_1(x_1) & -\dfrac{2æ\sqrt{\nu_2 s}}{\mu_2}\,\tilde{a}_1(a,s) & K_1(y_1) \end{vmatrix}, \tag{3.17}$$

$$D_3=\frac{1}{s\Delta_1}\begin{vmatrix} 0 & I_0(z_1) & -\tilde{h}_2 \\ I_0(x_1) & -I_0(y_1) & \tilde{h}_2-\tilde{h}_1 \\ \dfrac{\mu}{\sqrt{\nu}}\,I_1(x_1) & -I_1(y_1) & -\dfrac{2æ\sqrt{\nu_2 s}}{\mu_2}\,\tilde{a}_1(a,s) \end{vmatrix},$$

$$\Delta_1=\begin{vmatrix} 0 & I_0(z_1) & K_0(z_1) \\ I_0(x_1) & -I_0(y_1) & -K_0(y_1) \\ \dfrac{\mu}{\sqrt{\nu}}\,I_1(x_1) & -I_1(y_1) & K_1(y_1) \end{vmatrix},$$

where $x_1=a\sqrt{s/\nu_1}$; $y_1=a\sqrt{s/\nu_2}$; $z_1=b\sqrt{s/\nu_2}$; $\mu=\mu_1/\mu_2$; $\nu=\nu_1/\nu_2$. Since [4]

$$\begin{aligned} &\int_0^a rI_0\left(\sqrt{\frac{s}{\nu_1}}\,r\right)dr=\sqrt{\frac{\nu_1}{s}}\,aI_1\left(\sqrt{\frac{s}{\nu_1}}\,a\right), \\ &\int_a^b rI_0\left(\sqrt{\frac{s}{\nu_2}}\,r\right)dr=\frac{\nu_2}{s}\,[z_1I_1(z_1)-y_1I_1(y_1)], \\ &\int_a^b rK_0\left(\sqrt{\frac{s}{\nu_2}}\,r\right)dr=\frac{\nu_2}{s}\,[y_1K_1(y_1)-z_1K_1(z_1)]. \end{aligned} \tag{3.18}$$

Then taking into account (3.16), (3.18) and the second and third relations (3.15), we find

$$D_1 = -\frac{a\tilde{h}_1(s)}{2\sqrt{\nu_1 s}\, I_1(x_1)}\,,$$
$$D_2[z_1 I_1(z_1) - y_1 I_1(y_1)] + D_3[y_1 K_1(y_1) - z_1 K_1(z_1)] = -\frac{\tilde{h}_2(s)(b^2-a^2)}{2\nu_2}\,. \tag{3.19}$$

Substitution of D_1, D_2, D_3 from (3.17) into (3.19) allows us to define $\tilde{h}_1(s)$ and $\tilde{h}_2(s)$:

$$\tilde{h}_1(s) = \frac{F_1B_2 - A_2F_2}{A_1B_2 - A_2B_1}\,, \quad \tilde{h}_2(s) = \frac{A_1F_2 - F_1B_1}{A_1B_2 - A_2B_1}\,, \tag{3.20}$$

where

$$A_1 = \frac{x_1\Delta_1}{2I_1(x_1)} + I_0(z_1)K_1(y_1) + K_0(z_1)I_1(y_1),$$
$$A_2 = \frac{1}{y_1} - I_0(z_1)K_1(y_1) - I_1(y_1)K_0(z_1),$$
$$F_1 = \frac{2æ\tilde{a}_1(a,s)y_1}{a\rho_2}\,[I_0(y_1)K_0(z_1) - I_0(z_1)K_0(y_1)],$$
$$B_1 = \frac{\mu}{\sqrt{\nu}}\, I_1(x_1)\,[1 - y_1 I_1(y_1)K_0(z_1) - y_1 I_0(z_1)K_1(y_1)]\,,$$

$$B_2 = [z_1 I_1(z_1) - y_1 I_1(y_1)]\left[\frac{\mu}{\sqrt{\nu}}\, I_1(x_1)K_0(y_1) + I_0(x_1)K_1(y_1) - \right.$$
$$\left. - \frac{\mu}{\sqrt{\nu}}\, I_1(x_1)K_0(z_1)\right] + [y_1K_1(y_1) - z_1K_1(z_1)]\left[\frac{\mu}{\sqrt{\nu}}\, I_1(x_1)I_0(z_1) + \right.$$
$$\left. + I_0(x_1)I_1(y_1) - \frac{\mu}{\sqrt{\nu}}\, I_1(x_1)I_0(y_1)\right] + \frac{(1-\delta)}{2\delta}\, y_1^2\Delta_1,$$

$$F_2 = \frac{2æ\tilde{a}_1(a,s)}{a\rho_2}\, y_1 I_0(x_1)\,[1 - y_1K_0(z_1)I_1(y_1) - y_1K_1(y_1)I_0(z_1)]\,.$$

After some complicated mathematical treatment the limiting equalities

$$\lim_{s\to 0} s w_j(r,s) = w_j^s(r), \quad j = 1, 2,$$

are proved. This means that the solution tends to the stationary solution as time increases.

Figs. 1, 2 show the dimensionless function $\overline{w}_j = a^2 w_j/\nu_1$ for silicon-water system at temperature of 20^oC. Fig. 1 presents the case when $\alpha(\tau) = \sin(10^{-2}\tau)$, where $\tau = a^2t/\nu_1$ – is the dimensionless time that is $\alpha(\tau)$ does not have limit by $\tau \to \infty$. Thus, the solution with time growth does not converge to stationary. Fig. 2 shows the case when $\alpha(\tau) = 1 + e^{-\tau}\sin(\tau)$.

Remark 1. The problem of determination of the image $\tilde{b}_j(r,s)$ is similar to problem (3.2)–(3.4). One need to replace $-a_{j0}(r)$ with $-b_{j0}(r) - 2\tilde{a}_j(r,s)$ and $\tilde{\alpha}(s)$ with $\tilde{\beta}(s)$. Thus, these functions can be found with the use of (3.5)–(3.8).

The author is grateful his scientific advisor, doctor in physical and mathematical sciences V.K.Andreev for formulating the problem and for valuable advice. The work was supported by the Russian Foundation for Basic Research, grant 14-01-00067.

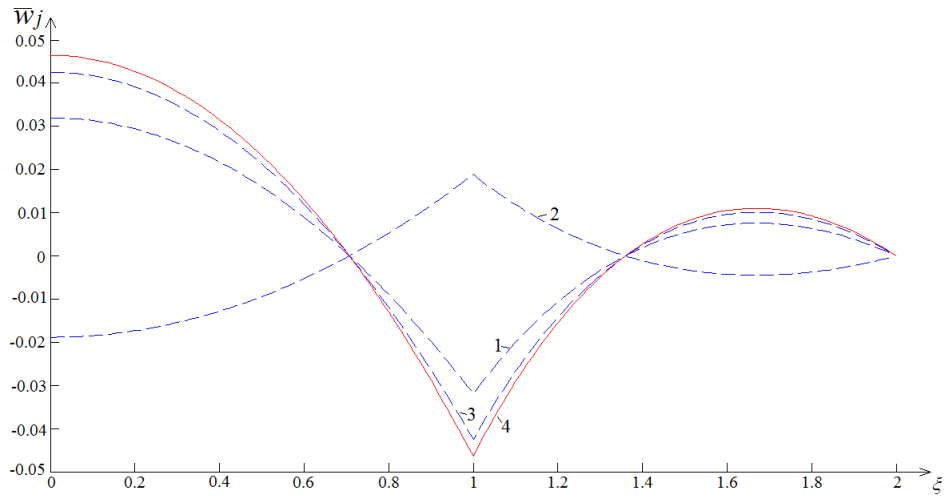

Fig. 1. Dimensionless function profiles $\overline{w}_j$ at $\alpha(\tau) = \sin(10^{-2}\tau)$; curve 1: $\tau = 200$; curve 2: $\tau = 400$; curve 3: $\tau = 700$; curve 4: stationary solution. Curves 1–3 with time growth does not converge to curve 4

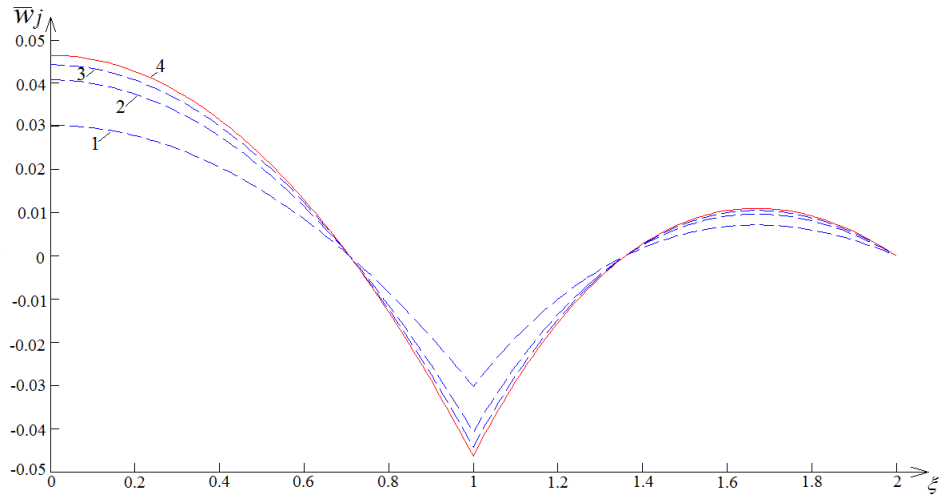

Fig. 2. Dimensionless function profiles $\overline{w}_j$ at $\alpha(\tau) = 1 + e^{-\tau}\sin(\tau)$; curve 1: $\tau = 20$; curve 2: $\tau = 50$; curve 3: $\tau = 100$; curve 4: stationary solution

References

[1] V.K.Andreev, Mathematical modeling of convective flows, study guide, Krasn. Gos. Univ., Krasnoyarsk, 2006 (in Russian).

[2] L.V.Ovsyannikov, Group analysis of differential equations, Academic Press, New York, 1982.

[3] M.A.Lavrentiev, B.V.Shabat, Methods of the theory of functions of a complex variable,

Nauka, Moscow, 1973 (in Russian).

[4] G.Bateman, A.Erdelyi, Higher transcendental functions. 2. Bessel functions, the parabolic cylinder function, orthogonal polynomials, McGraw-Hill, 1953.

Осесимметрическое термокапиллярное движение в цилиндре при малых числах Марангони

Евгений П. Магденко

В статье решена линейная задача об осесимметрическом термокапиллярном движении двух несмешивающихся вязких теплопроводных жидкостей в цилиндрической трубе. Их общая поверхность раздела фиксируема и недеформируема. Задача является обратной, так как градиенты давлений есть искомые функции. В изображениях по Лапласу решения находятся в виде квадратур. Доказано, что если температура на стенке трубы стабилизируется со временем, то решение также с ростом времени стремится к стационарному режиму. Проведённые численные расчёты хорошо соотносятся с теоретическими результатами.

Ключевые слова: термокапиллярность, обратная задача, преобразование Лапласа, поверхность раздела.