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On the Boussinesq Approximation in the Problems of Convection Induced by High–Frequency Vibration

Ilya I. Ryzhkov∗

Institute of Computational Modelling SB RAS,
Krasnoyarsk, 660036
Russia

Yuri A. Gaponenko

Institute of Computational Modelling SB RAS,
Krasnoyarsk, 660036
Russia
Microgravity Research Center, Université Libre de Bruxelles,
CP 165/62, av. F.D.Roosevelt 50, B–1050 Brussels,

Belgium

The applicability of Boussinesq approximation to the problems of thermovibrational convection in closed volumes is analyzed. The limit of high frequency and small amplitude is considered on the basis of averaging approach. The magnitudes of oscillatory and averaged flow fields are estimated. It is found that the dependence of the Reynolds number for averaged motion on the ratio of Gershuni and Prandtl numbers obeys linear and square root laws for small and large Reynolds numbers, respectively. It provides new essential information about the intensity of averaged flows in a wide range of vibration stimuli. Taking into account the obtained estimations, the basic assumptions of Boussinesq approach are applied to the momentum, continuity, and energy equations for a compressible, viscous heat–conducting fluid. The contribution of viscous energy dissipation and pressure work to the energy balance is also taken into account. The order of magnitude analysis provides a number of dimensionless parameters, the smallness of which guarantees the validity of Boussinesq approximation

Keywords: Boussinesq approximation, vibrational convection, averaged motion.

Introduction

The application of vibrations to a fluid system with density gradient can cause relative flows inside the fluid. If this gradient results from thermal or compositional variations, such flows are known as thermovibrational or solutovibrational convection, respectively. The case of small amplitude and high frequency vibration (when the period is much smaller than the characteristic viscous and heat (mass) diffusion times) is of special interest. In this case, the flow field can be represented as a superposition of 'quick' part, which oscillates with the frequency of vibration, and 'slow' time–averaged part, which describes the non–linear response of the fluid to a periodic excitation [1]. In the theory of convection, the averaging approach was first used in [2]. The mathematical jusitfication of this approach for convection problems was given in [3].

The study of vibrational impact on fluids has fundamental and applied importance. Vibrational convection provides a mechanism of heat and mass transfer due to the existence of averaged flows. Such flows show some similarity with gravity–induced convection and might serve as a way to control and operate fluids in space [4]. High–frequency oscillations (g–jitter) onboard

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∗rii@icm.krasn.ru

microgravity platforms can disturb the experiments that require purely diffusive heat and mass transfer: crystal growth, measurement of transport coefficients, etc. [5].

There have been extensive *theoretical* studies of thermovibrational convection in weightlessness and ground conditions. The fundamental treatise [1] comprises a systematic study of convective flows induced by high and finite frequency vibrations in closed and infinite cavities. Thermovibrational convection in square, rectangular, and cubic cavities was widely investigated providing a variety of averaged flow structures and bifurcation scenarios [6–8]. Influence of vibration on double diffusive convection with the Soret effect was analyzed in [9]. The *experimental* studies of vibrational impact on fluids were performed in a number of works. The considered configurations include vertical and horizontal layers [10], Hele–Shaw cell [11], etc. The influence of vibration on the propagation of heat from a point-like source in microgravity conditions was investigated in [12]. Recent parabolic flight experiments [13–15] provided one of the first quantitative observation of averaged flows and related heat transfer in microgravity environment.

The most part of existing theoretical and numerical studies of vibrational convection were performed in the frame of Boussinesq approximation. The fluid was considered as incompressible and the density variations were taken into account only in the vibrational force term. It should be noted that the applicability of this approach in gravitational convection problems was extensively studied in a number of works [16–19]. Concerning vibrational convection problems, the justification of Boussinesq approach was discussed in [1]. However, in that paper, the contribution of pressure work and viscous energy dissipation to the energy balance was not taken into account. This contribution can become significant at high frequencies. In addition, when the frequency of vibration is increased, the pressure gradients induced by the vibrational force can become so large that the fluid cannot be considered incompressible anymore. In this respect, a complete set of criteria, which describe the validity domain of Boussinesq approximation in vibrational convection problems, is still necessary.

In this paper, we analyze the applicability of Boussinesq approximation to the problems of thermovibrational convection in closed volumes. The study is focused on the limit of high frequency and small amplitude. It is assumed that no other external forces except vibration are present. In the first part, a rigorous derivation of averaged equations is given. Note that the averaging procedure is not straightforward; furthermore, it is not described in sufficient detail in the related literature [1,2]. In the second part, the criteria for the validity of Boussinesq approach are derived. The basic assumptions of this approach are applied to the set of momentum, continuity, and energy equations for a compressible, viscous heat–conducting fluid, where the contribution of viscous energy dissipation and pressure work to the energy balance is taken into account. Two cases, which correspond to averaged flows with small and large Reynolds numbers and provide different estimations of averaged velocity, are considered. The obtained estimations are confirmed by numerical modelling. The order of magnitude analysis provides a number of dimensionless parameters, the smallness of which guarantees the validity of Boussinesq approximation. The main results are summarized in Conclusion.

1.   Governing Equations and Boussinesq Criteria

Consider compressible viscous heat–conducting fluid in a closed cavity. It is assumed that the cavity performs translational harmonic oscillations with the angular frequency ω and amplitude A in the direction of vector $\boldsymbol{e}$. There are no other external forces (such as, for example, gravity) imposed on the fluid. The general equations describing the fluid in the coordinate system associated with the cavity have the form [1, 19, 20]:

$$\rho\left(\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\,\boldsymbol{V}\right) = -\nabla P + \mu \nabla^2 \boldsymbol{V} + \left(\lambda + \frac{\mu}{3}\right) \nabla(\nabla \cdot \boldsymbol{V}) + \rho A \omega^2 \cos(\omega t)\,\boldsymbol{e}, \qquad (1)$$

$$\partial_t \rho + \boldsymbol{V} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{V} = 0, \tag{2}$$

$$\rho c_p \big(\partial_t T + \boldsymbol{V} \cdot \nabla T\big) - \beta_T (T_0 + T)\big(\partial_t P + \boldsymbol{V} \cdot \nabla P\big) =$$

$$= \kappa \nabla^2 T + \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \boldsymbol{V} \right)^2 + \lambda (\nabla \cdot \boldsymbol{V})^2. \tag{3}$$

Here $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{V} = (V_1, V_2, V_3)$ are the coordinate and velocity vectors, T and P are the deviations of full temperature and pressure from the mean values T_0 and P_0, respectively, ρ is the fluid density, μ is the dynamic viscosity, λ is the second viscosity, c_p is the specific heat capacity, κ is the thermal conductivity, β_T is the thermal expansion coefficient, and δ_{ij} is the Kronecker delta. Note that the governing equations are written for the case of constant physical properties μ, λ, κ, c_p for simplicity (this requirement will be introduced below).

Let us now formulate the basic assumptions of Boussinesq approximation.

1. The deviations of temperature and pressure from the mean values are small, i.e.

$$|T/T_0| \ll 1, \qquad |P/P_0| \ll 1. \tag{4}$$

Then the equation of state can be written as

$$\rho = \rho_0 (1 - \beta_T T + \beta_P P),$$

where $\beta_T = -\rho^{-1} \partial \rho / \partial T$ and $\beta_P = \rho^{-1} \partial p / \partial T$ (the latter is the isothermal compressibility).

2. The density variation with pressure is much smaller than that with temperature, and the latter is also small:

$$|\beta_P P| \ll |\beta_T T| \ll 1. \tag{5}$$

3. The variation of density with T and P is negligible everywhere except the external force term in Eq. (1), where the density is written as

$$\rho = \rho_0 (1 - \beta_T T). \tag{6}$$

4. The contribution of viscous energy dissipation and pressure variations to the energy balance (Eq. (3)) is negligible.

5. The variations of temperature and pressure are sufficiently small to assume the constancy of physical properties μ, λ, κ, c_p.

Under these assumptions, equations (1)–(3) are reduced to the system describing the motion of incompressible fluid under vibration:

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla) \, \boldsymbol{V} = -\rho_0^{-1} \nabla P + \nu \nabla^2 \boldsymbol{V} + (1 - \beta_T T) A \omega^2 \cos(\omega t) \, \boldsymbol{e}, \tag{7}$$

$$\partial_t T + \boldsymbol{V} \cdot \nabla T = \chi \nabla^2 T, \tag{8}$$

$$\nabla \cdot \boldsymbol{V} = 0, \tag{9}$$

where $\nu = \mu/\rho_0$ is the kinematic viscosity and $\chi = \kappa/\rho_0 c_p$ is the thermal diffusivity.

In this work, we consider vibrations with high frequency and small amplitude (the exact definitions will be given below). Then the method of averaging can be effectively applied for studying vibrational convective flows [1]. Using this method, we will estimate the magnitudes of velocity, pressure, and temperature fields and derive the criteria, under which assumptions 1–5 are satisfied.

2.   Averaging Approach in Vibrational Convection Problems

2.1. Derivation of Averaged Equations

In this section, we provide a detailed derivation of averaged equations for vibrational convective flows. The description of this procedure in the existing literature [1, 2] lacks a number of important details, which will be pointed out and extensively discussed below. We believe that this part will be useful for those readers, who would like to acquaint themselves with averaging approach in convection problems.

Let us denote the period of vibration by $\tau = 2\pi/\omega$. In the averaging method, each field is decomposed into "slow" time–averaged part (with characteristic time much larger than τ) and "fast" oscillatory part (with characteristic time τ):

$$\boldsymbol{V} = \overline{\boldsymbol{V}} + \boldsymbol{V}', \qquad T = \overline{T} + T', \qquad \widetilde{P} = \overline{P} + P'. \tag{10}$$

Here $\widetilde{P}$ is the modified pressure, which is given by

$$\widetilde{P} = P - \rho_0 A \omega^2 \cos(\omega t)\, \boldsymbol{e} \cdot \boldsymbol{x}. \tag{11}$$

For a given function of time and space coordinates $f(t, \boldsymbol{x})$, the averaged and oscillatory components are defined by

$$\overline{f}(t, \boldsymbol{x}) = \frac{1}{\tau} \int\limits_{t-\tau/2}^{t+\tau/2} f(\tau', \boldsymbol{x})\, d\tau', \tag{12}$$

$$f'(t, \boldsymbol{x}) = f(t, \boldsymbol{x}) - \overline{f}(t, \boldsymbol{x}). \tag{13}$$

Further, a bar over the quantity will indicate the averaging operator according to (12). If $\overline{f}$ does not depend on time, then the averaged motion is called *stationary*, while the full motion is called *quasi–stationary*. It follows from (12) that

$$\overline{\partial_t f} = \partial_t \overline{f}, \qquad \overline{\nabla f} = \nabla \overline{f}. \tag{14}$$

We assume that the variation of an averaged quantity with time during the vibration period τ is negligible, so the second averaging coincides with the original averaged quantity:

$$\overline{\overline{f}} = \overline{f}. \tag{15}$$

It follows from (13) and (15) that the oscillatory component has zero average:

$$\overline{f'} = 0. \tag{16}$$

We also suppose that

$$\overline{\overline{f}\, h} = \overline{f}\, \overline{h}, \tag{17}$$

where $h(t, \boldsymbol{x})$ is a "fast" oscillatory function of t (with characteristic time τ). For quasi–stationary motion, relation (17) follows from (12). Otherwise, it shoud be postulated as an additional property of the averaging operator [21].

Let us now apply definitions (12) and (13) to the velocity, pressure, and temperature fields taking into account (14)–(17). Substituting representation (10) into the governing equations (7)–(9) gives

$$\partial_t \overline{\boldsymbol{V}} + \partial_t \boldsymbol{V}' + (\overline{\boldsymbol{V}} \cdot \nabla)\, \overline{\boldsymbol{V}} + (\overline{\boldsymbol{V}} \cdot \nabla)\, \boldsymbol{V}' + (\boldsymbol{V}' \cdot \nabla)\, \overline{\boldsymbol{V}} + (\boldsymbol{V}' \cdot \nabla)\, \boldsymbol{V}' =$$

$$= -\rho_0^{-1}\nabla\overline{P} - \rho_0^{-1}\nabla P' + \nu\nabla^2\overline{\boldsymbol{V}} + \nu\nabla^2\boldsymbol{V}' - \beta_T(\overline{T} + T')A\omega^2\cos(\omega t)\,\boldsymbol{e}, \tag{18}$$

$$\partial_t\overline{T} + \partial_t T' + \overline{\boldsymbol{V}}\cdot\nabla\overline{T} + \overline{\boldsymbol{V}}\cdot\nabla T' + \boldsymbol{V}'\cdot\nabla\overline{T} + \boldsymbol{V}'\cdot\nabla T' = \chi\nabla^2\overline{T} + \chi\nabla^2 T', \tag{19}$$

$$\nabla\cdot\overline{\boldsymbol{V}} + \nabla\cdot V' = 0. \tag{20}$$

Applying the averaging operator to equations (18)–(19), we obtain the equations for averaged fields

$$\partial_t\overline{\boldsymbol{V}} + (\overline{\boldsymbol{V}}\cdot\nabla)\,\overline{\boldsymbol{V}} + \overline{(\boldsymbol{V}'\cdot\nabla)\,\boldsymbol{V}'} = -\rho_0^{-1}\nabla\overline{P} + \nu\nabla^2\overline{\boldsymbol{V}} - \overline{\beta_T T'A\omega^2\cos(\omega t)}\,\boldsymbol{e}, \tag{21}$$

$$\partial_t\overline{T} + \overline{\boldsymbol{V}}\cdot\nabla\overline{T} + \overline{\boldsymbol{V}'\cdot\nabla T'} = \chi\nabla^2\overline{T}, \tag{22}$$

$$\nabla\cdot\overline{\boldsymbol{V}} = 0. \tag{23}$$

One can see that this system differs from the classical Navier–Stokes and heat trasnfer equations by the terms containing averaged products of oscillatory components. Let us now subtract the averaged equations (21)–(23) from the full equations (18)–(20). It leads to the system for oscillatory components of motion:

$$\partial_t\boldsymbol{V}' + (\overline{\boldsymbol{V}}\cdot\nabla)\,\boldsymbol{V}' + (\boldsymbol{V}'\cdot\nabla)\,\overline{\boldsymbol{V}} + (\boldsymbol{V}'\cdot\nabla)\,\boldsymbol{V}' - \overline{(\boldsymbol{V}'\cdot\nabla)\,\boldsymbol{V}'} =$$

$$= -\rho_0^{-1}\nabla P' + \nu\nabla^2\boldsymbol{V}' - \beta_T(\overline{T} + T')A\omega^2\cos(\omega t)\,\boldsymbol{e} + \overline{\beta_T T'A\omega^2\cos(\omega t)}\,\boldsymbol{e}, \tag{24}$$

$$\partial_t T' + \overline{\boldsymbol{V}}\cdot\nabla T' + \boldsymbol{V}'\cdot\nabla\overline{T} + \boldsymbol{V}'\cdot\nabla T' - \overline{\boldsymbol{V}'\cdot\nabla T'} = \chi\nabla^2 T', \tag{25}$$

$$\nabla\cdot\boldsymbol{V}' = 0. \tag{26}$$

Note that in the existing literature [1, 2], the equations for oscillatory components are obtained by selecting "fast" terms from equations (18)–(19) and equating the result to zero. The obtained system does not contain averaged terms (such as $\overline{(\boldsymbol{V}'\cdot\nabla)\,\boldsymbol{V}'}$), which are present in equations (24)–(26). This approach is incorrect from mathematical point of view.

In what follows, we will need the scales of time, length, and averaged temperature, which are naturally determined by the vibration period τ, typical size of the system L, and the applied temperature difference ΔT, respectively. Concerning the scale of averaged velocity, we note that it must depend on the density inhomogeniety caused by the applied temperature difference (i.e. $\beta_T\Delta T$), the amplitude and frequency of vibration, and viscous properties of the fluid. Here we assume that $|\overline{\boldsymbol{V}}|$ does not exceed ν/L by the order of magnitude. This choice will be discussed in details in Section 4.1. The scales of oscillatory components $\boldsymbol{V}'$, P', T' are not known in advance and will be determined later. Note that in the existing derivations [1, 2], the authors implicitly assume that $|\overline{\boldsymbol{V}}| \sim \nu/L$ without any justification of such choice.

Equations (24)–(26) can be simplified by introducing several assumptions.

1. The period of vibration is much smaller than the reference viscous and thermal times:

$$\tau \ll \min(L^2/\nu,\, L^2/\chi). \tag{27}$$

It allows us to neglect the terms $(\overline{\boldsymbol{V}}\cdot\nabla)\,\boldsymbol{V}'$, $(\boldsymbol{V}'\cdot\nabla)\,\overline{\boldsymbol{V}}$, $\nu\nabla^2\boldsymbol{V}'$ in comparison with $\partial_t\boldsymbol{V}'$ in equation (24) and the terms $\overline{\boldsymbol{V}}\cdot\nabla T'$ and $\chi\nabla^2 T'$ with respect to $\partial_t T'$ in equation (25).

2. The displacement of oscillating fluid particles is much smaller than the characteristic length scale:

$$\tau|\boldsymbol{V}'| \ll L. \tag{28}$$

It follows that the convective terms $(\boldsymbol{V}'\cdot\nabla)\,\boldsymbol{V}'$ and $\boldsymbol{V}'\cdot\nabla T'$ are negligible in comparison with $\partial_t\boldsymbol{V}'$ and $\partial_t T'$, respectively. The same is true for the average of the above products, as the latter

cannot exceed the original non–averaged terms by magnitude. Note that in the literature [1], the present assumption is introduced in the form of its consequence (41), which hides the real physical meaning expressed by (28).

3. The oscillations of temperature field are much smaller than the applied temperature difference: $|T'| \ll \Delta T$, so the last two terms containing T' can be neglected in equation (24).

Under the above assumptions, equations for the oscillatory fields become

$$\partial_t \boldsymbol{V}' = -\rho_0^{-1} \nabla P' - \beta_T \overline{T} A \omega^2 \cos(\omega t) \boldsymbol{e}, \tag{29}$$

$$\partial_t T' = -\boldsymbol{V}' \cdot \nabla \overline{T}, \tag{30}$$

$$\nabla \cdot \boldsymbol{V}' = 0. \tag{31}$$

To solve equation (29), the vector $\overline{T} \boldsymbol{e}$ is decomposed into the solenoidal $\mathbf{W}$ and irrotational $\nabla \Phi$ parts:

$$\overline{T} \boldsymbol{e} = \mathbf{W} + \nabla \Phi, \tag{32}$$

where

$$\nabla \cdot \mathbf{W} = 0, \qquad \mathbf{W} \cdot \boldsymbol{n}\big|_\Gamma = 0, \tag{33}$$

Here $\boldsymbol{n}$ is the unit normal vector to the impermeable boundary Γ of the domain with fluid. The above decomposition exists for a wide class of vector fields $\overline{T} \boldsymbol{e}$ [3]. Substituting (32) into (29), we find

$$\partial_t \boldsymbol{V}' + \beta_T A \omega^2 \cos(\omega t) \mathbf{W} = -\rho_0^{-1} \nabla P' - \beta_T A \omega^2 \cos(\omega t) \nabla \Phi. \tag{34}$$

The right–hand side of this equation represents irrotational vector field, thus the left–hand side has the same property. Then it can be represented as a gradient of some function

$$\partial_t \boldsymbol{V}' + \beta_T A \omega^2 \cos(\omega t) \mathbf{W} = \nabla \Psi. \tag{35}$$

It follows from (31) and (33) that $\nabla^2 \Psi = 0$. To proceed further, one needs to specify boundary condition for the oscillatory velocity component. Note that the viscous force driving the oscillatory flow has been neglected when deriving equation (29), see Assumption 1. It means that the existence of Stokes boundary layer for the oscillatory flow near the rigid impermeable boundary is not taken into account. So, the non–permeability condition rather than the no–slip one should be imposed on the oscillatory velocity component:

$$\boldsymbol{V}' \cdot \boldsymbol{n}\big|_\Gamma = 0. \tag{36}$$

Taking into account (33) and (35), we arrive to the problem $\nabla^2 \Psi = 0$, $\nabla \Psi \cdot \boldsymbol{n}\big|_\Gamma = 0$, from which it follows that $\nabla \Psi = 0$ [22]. Then the left and right–hand sides of (34) are equal to zero, so

$$\partial_t \boldsymbol{V}' = -\beta_T A \omega^2 \cos(\omega t) \mathbf{W}, \qquad \rho_0^{-1} \nabla P' = -\beta_T A \omega^2 \cos(\omega t) \nabla \Phi. \tag{37}$$

Note that the separation of (34) into the above equations becomes possible only after the boundary conditions on $\mathbf{W}$ and $\boldsymbol{V}'$ are specified by (33) and (36), respectively. This logical order is not followed in the derivation presented in [1], where the solenoidal and irrotational parts of (34) are separately equalized to zero with any further explanation.

Let us now integrate the first equation in (37) from t_0 to t, where t_0 is fixed and $|t - t_0| \leqslant \tau$. Taking into account that $\mathbf{W}$ practically does not change with time during the vibration period τ, we obtain $\boldsymbol{V}'(t) - \boldsymbol{V}'(t_0) = -\beta_T A \omega \mathbf{W} \big(\sin(\omega t) - \sin(\omega t_0)\big)$. In this relation, one should put $\boldsymbol{V}'(t_0) = -\beta_T A \omega \mathbf{W} \sin(\omega t_0)$ in order for $\boldsymbol{V}'$ to have a zero average, see (16). The obtained expression for $\boldsymbol{V}'$ is substituted into (30), and the resulting equation is integrated similarly. Further, P' is found by integrating the last equation in (37) over the space coordinates. Note that

P' is determined accurate to an arbitrary function of time with zero average. This ambiguity can be removed by specifying the full pressure at some point of the domain. Finally, the expressions for the oscillatory components are written as

$$\boldsymbol{V}' = -\beta_T A\omega \sin(\omega t)\mathbf{W}, \qquad T' = -\beta_T A \cos(\omega t)\mathbf{W} \cdot \nabla \overline{T}, \tag{38}$$

$$P' = -\rho_0 \beta_T A\omega^2 \cos(\omega t)\, \Phi. \tag{39}$$

It follows from (32) that $|\mathbf{W}| \sim |\nabla\Phi| \sim \Delta T$, which allows us to determine the characteristic scales of oscillatory fields:

$$|\boldsymbol{V}'| \sim \beta_T \Delta T A\omega, \qquad |T'| \sim \beta_T \Delta T^2 A/L, \qquad |P'| \sim \rho_0 \beta_T \Delta T A\omega^2 L. \tag{40}$$

Then the criteria introduced in Assumptions 2 (see equation (28)) and 3 become

$$A \ll \frac{L}{\beta_T \Delta T}. \tag{41}$$

It provides a limitation on the amplitude range.

The final equations for averaged fields are obtained by substituting expressions (38) and (39) into (21)–(23) and averaging the related terms:

$$\partial_t \overline{\mathbf{V}} + (\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}} = -\rho_0^{-1}\nabla\overline{P} + \nu\nabla^2\overline{\mathbf{V}} + \frac{(\beta_T A\omega)^2}{2}\big((\overline{T}\boldsymbol{e} - \nabla\Phi) \cdot \nabla\big)\nabla\Phi, \tag{42}$$

$$\partial_t \overline{T} + (\overline{\mathbf{V}} \cdot \nabla)\overline{T} = \chi\nabla^2\overline{T}, \tag{43}$$

$$\nabla \cdot \overline{\mathbf{V}} = 0, \tag{44}$$

$$\nabla^2\Phi - \nabla\overline{T} \cdot \boldsymbol{e} = 0. \tag{45}$$

Here the last equation follows from (32). The latter relation was also taken into account when deriving equation (42).

2.2. Thermovibrational Convection in a Square Cavity

In this paper, we perform numerical simulations of thermovibrational convection, which are complementary to the theoretical results. This study is required for validating the theoretical estimations of velocity, temperature, and pressure magnitudes. These estimations will be used for deriving the criteria for the validity of Boussinesq approximation.

Let us consider a square cavity with rigid impermeable walls of the size L (Fig. 1). The top and bottom walls are kept at constant temperatures T_{hot} and T_{cold}, respectively, providing the temperature difference $\Delta T = T_{hot} - T_{cold}$. On the lateral walls, the linear temperature profile is imposed. The cavity performs translational harmonic oscillations with the frequency ω and amplitude A in the X direction, which is perpendicular to the temperature gradient. On the boundaries of the cavity, the no–slip condition for the average velocity is imposed. The boundary condition for function Φ is derived from (32) and (33):

$$(\nabla\Phi - \overline{T}\boldsymbol{e}) \cdot \boldsymbol{n}\big|_\Gamma = 0.$$

Now let us introduce the dimensionless variables

$$\overline{\mathbf{v}} = \frac{\overline{\mathbf{V}}}{\nu/L}, \qquad \overline{\Theta} = \frac{\overline{T}}{\Delta T}, \qquad \overline{p} = \frac{\overline{P}}{\rho_0 \nu^2/L^2}, \qquad \varphi = \frac{\Phi}{\Delta T L}.$$
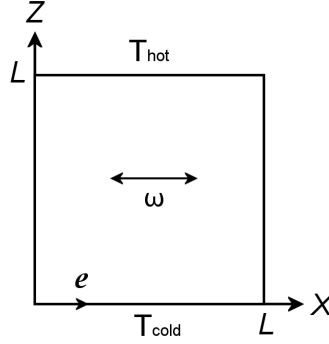
Fig. 1. Square cavity and coordinate system

The scales of length and time are taken as L and L^2/ν, respectively. The averaged equations will be solved in terms of stream function ψ and vorticity ζ, which are introduced by the formulas

$$\overline{\mathbf{v}} = (u, w) = \left(\frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x} \right), \qquad \zeta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}.$$

The governing equations (42)–(45) are rewritten in the form

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial z} = \nabla^2 \zeta + \frac{\mathrm{Gs}}{\mathrm{Pr}} \left(\frac{\partial \overline{\Theta}}{\partial z} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \overline{\Theta}}{\partial x} \frac{\partial^2 \varphi}{\partial x \partial z} \right), \tag{46}$$

$$\frac{\partial \overline{\Theta}}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial \overline{\Theta}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \overline{\Theta}}{\partial z} = \frac{1}{\mathrm{Pr}} \nabla^2 \overline{\Theta}, \tag{47}$$

$$\nabla^2 \varphi = \frac{\partial \overline{\Theta}}{\partial x}, \tag{48}$$

$$\nabla^2 \psi = -\zeta, \tag{49}$$

$$\nabla^2 \overline{p} = 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial z^2} - 2 \left(\frac{\partial^2 \psi}{\partial x \partial z} \right)^2 + \frac{\mathrm{Gs}}{\mathrm{Pr}} \left[-\left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2 - 2 \left(\frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 \varphi}{\partial y^2} \right)^2 + \right.$$

$$\left. + \frac{\partial \overline{\Theta}}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \overline{\Theta}}{\partial z} \frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \overline{\Theta}}{\partial x^2} \left(\overline{\Theta} - \frac{\partial \varphi}{\partial x} \right) - \frac{\partial^2 \overline{\Theta}}{\partial x \partial z} \frac{\partial \varphi}{\partial z} \right]. \tag{50}$$

The system includes the dimensionless Prandtl and Gershuni numbers:

$$\mathrm{Pr} = \frac{\nu}{\chi}, \qquad \mathrm{Gs} = \frac{(A \omega \beta_T \Delta T L)^2}{2 \nu \chi}.$$

The Gershuni number (also known as the vibrational Rayleigh number) can be regarded as the ratio of mean vibrational buoyancy force to the product of momentum and thermal diffusivities. It describes the vibrational mechanism of convection represented by the mean flow.

The boundary conditions of the problem are written as

$$x = 0, 1: \quad \psi = \frac{\partial \psi}{\partial x} = 0, \quad \overline{\Theta} = z, \qquad \frac{\partial \varphi}{\partial x} = \overline{\Theta}, \qquad \frac{\partial \overline{p}}{\partial x} = \frac{\partial^3 \psi}{\partial x^2 \partial z} - \frac{\mathrm{Gs}}{\mathrm{Pr}} \frac{\partial \overline{\Theta}}{\partial z} \frac{\partial \varphi}{\partial z},$$

$$z = 0, 1: \quad \psi = \frac{\partial \psi}{\partial z} = 0, \quad \overline{\Theta} = 0, 1, \quad \frac{\partial \varphi}{\partial z} = 0, \qquad \frac{\partial \overline{p}}{\partial z} = -\frac{\partial^3 \psi}{\partial x \partial z^2}, \tag{51}$$

while the initial conditions have the form

$$t = 0: \qquad \psi = \zeta = \overline{p} = 0, \qquad \overline{\Theta} = z, \qquad \frac{\partial \varphi}{\partial x} = \overline{\Theta}, \qquad \frac{\partial \varphi}{\partial z} = 0. \tag{52}$$

We assume that the non–uniform thermal field (linear temperature profile) is established before vibration is applied. The last two conditions follow from (32), where the amplitude of oscillatory velocity component $\mathbf{W} = 0$ at the initial moment of time. It should be noted that in the considered problem, the pressure is determined up to some arbitrary constant. To specify this constant, the minimal pressure in the volume is set to zero (it is enough for our purposes as we are interested in the pressure variation across the volume).

The problem (46)–(52) is solved by a finite–difference method using a regular equally spaced mesh $[101 \times 101]$. The dimensionless time step was 10^{-5}. The time derivatives are forward differenced and the convective and diffusive terms are central approximated. The Poisson equation (50) for pressure $\overline{p}$ is solved by introducing an artificial iterative term, which is analogous to the time-derivative one. ADI method is used to solve the time-dependent problem for the stream function, vorticity, temperature, pressure, and the amplitude of fast pressure φ. More details about the numerical procedure can be found in [23].

3. Boussinesq Approximation for Vibrational Convection Problems

3.1. Estimation of Averaged Flow Fields

To derive the criteria, under which Boussinesq assumptions 1–5 (see Section 1) are satisfied, we need to estimate the magnitudes of velocity, temperature, and pressure. The characteristic scales of oscillatory fields are given by (40). The time derivatives of oscillatory components are estimated by differentiating (38) and (39) with respect to "fast" time t (note that the functions $\overline{T}$, $\mathbf{W}$, Φ do not depend on this time).

The averaged fields should be estimated from equations (42)–(45). Let us first consider the case of fully developed stationary mean flow. Then the averaged temperature has the order of the applied temperature difference: $|\overline{T}| \sim \Delta T$. To estimate the averaged velocity, we note that convective or viscous term (or both) must be of the same order of magnitude as the mean vibrational force term in equation (42). In addition, these two terms cannot become large in comparison with the external force term as the latter is the source of motion.

Let us first suppose that

$$|\nu \nabla^2 \overline{\mathbf{V}}| \sim \left| \frac{(\beta_T A \omega)^2}{2} \left((\overline{T} \boldsymbol{e} - \nabla \Phi) \cdot \nabla \right) \nabla \Phi \right|. \tag{53}$$

It gives the following magnitude of averaged velocity:

$$|\overline{\mathbf{V}}| \sim \frac{(\beta_T \Delta T A \omega)^2 L}{\nu}. \tag{54}$$

Comparing the magnitude of convective and viscous terms, we find

$$\frac{|(\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}}|}{|\nu \nabla^2 \overline{\mathbf{V}}|} \sim \frac{|\overline{\mathbf{V}}|L}{\nu} \equiv \mathrm{Re} \sim \frac{\mathrm{Gs}}{\mathrm{Pr}}. \tag{55}$$

where Re is the Reynolds number. This estimation is valid for small Reynolds numbers (Re < 1). When Re is large, relation (55) predicts that viscous term is much smaller than the convective

one. It is in contradiction with the original assumption that viscous term is comparable with the mean vibrational force term.

Now consider the case when the convective term is comparable with the mean vibrational force term:

$$|(\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}}| \sim \left| \frac{(\beta_T A \omega)^2}{2} ((\overline{T}\boldsymbol{e} - \nabla\Phi) \cdot \nabla)\nabla\Phi \right|. \tag{56}$$

It provides the following estimation

$$|\overline{\mathbf{V}}| \sim \beta_T \Delta T A \omega. \tag{57}$$

Comparison of convective and viscous terms gives

$$\frac{|(\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}}|}{|\nu\nabla^2\overline{\mathbf{V}}|} \sim \frac{|\overline{\mathbf{V}}|L}{\nu} \equiv \mathrm{Re} \sim \left(\frac{\mathrm{Gs}}{\mathrm{Pr}}\right)^{1/2}, \tag{58}$$

Arguing in the same way as above, we conclude that the obtained estimation of velocity is valid for large Reynolds numbers ($\mathrm{Re} \geqslant 1$).

To validate estimations (54) and (57), numerical calculations of steady vibrational mean flows in a square cavity have been performed. The calculations have shown that one should introduce a proportionality factor between the Reynolds number and the ratio Gs/Pr in relations (55) and (58) in order to get a correct estimation of averaged velocity. For $\mathrm{Re} < 1$, we have found that

$$\mathrm{Re} \sim \gamma^2 \frac{\mathrm{Gs}}{\mathrm{Pr}}, \qquad |\overline{\mathbf{V}}| \sim \frac{(\gamma\beta_T \Delta T A \omega)^2 L}{\nu}, \tag{59}$$

while for $\mathrm{Re} \geqslant 1$

$$\mathrm{Re} \sim \gamma \left(\frac{\mathrm{Gs}}{\mathrm{Pr}}\right)^{1/2}, \qquad |\overline{\mathbf{V}}| \sim \gamma\beta_T \Delta T A \omega. \tag{60}$$

One can see that for large Reynolds numbers, the proportionality factor γ represents the ratio of averaged and oscillatory velocity components: $\gamma \sim |\overline{\mathbf{V}}|/|\boldsymbol{V}'|$, see also (40). The averaged velocity is normally smaller than the oscillatory one, so one can expect that $0 < \gamma < 1$ in the general case.

The obtained dependence of the Reynolds number on the ratio Gs/Pr is presented in Fig. 2a for $\mathrm{Pr} = 25$ (the case of liquid). The results of numerical calculations are shown by points, while the linear and square laws are represented by dashed and solid lines, respectively. The values of Reynolds number obtained from numerical simulation correspond to maximum averaged velocity in the cavity. The proportionality factor is $\gamma = 0.0195$. The results provide a good agreement between numerical calculations and suggested linear and square root laws. It validates estimations (59) and (60).

According to Fig. 2a, the order of the Reynolds number does not exceed unity in a wide range of Gershuni numbers ($0 \leqslant \mathrm{Gs} \leqslant 200 \times 10^3$). It means that the order of averaged velocity does not exceed ν/L (this assumption was introduced when deriving the equations for oscillatory fields in Section 3.1). The average velocity increases with increasing the Gershuni number. However, for very large values of Gs, the steady state motion transforms into non–damped oscillations and eventually chaotic regime [1].

To estimate the averaged pressure, we note that the pressure gradient term should be comparable with the vibrational force term in equation (42):

$$|\rho_0^{-1}\nabla\overline{P}| \sim \left| \frac{(\beta_T A \omega)^2}{2} ((\overline{T}\boldsymbol{e} - \nabla\Phi) \cdot \nabla)\nabla\Phi \right|.$$

It gives $|\overline{P}| \sim \rho_0(\beta_T \Delta T A \omega)^2$, which is equivalent to $|\overline{p}| \sim \varepsilon \frac{\mathrm{Gs}}{\mathrm{Pr}}$ in dimensionless form. Here we have introduced a proportionality factor ε, which can be determined from numerical simulation.
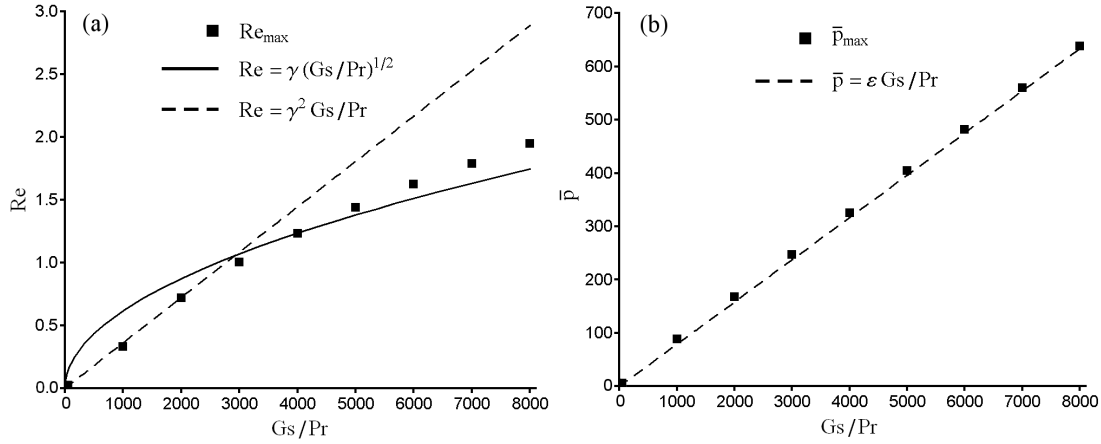
Fig. 2. The dependence of the Reynolds number (a) and dimensionless pressure (b) on the ratio Gs/Pr. The results of numerical calculations are shown by points, while the linear and square laws are represented by dashed and solid lines, respectively. The proportionality factors are $\gamma = 0.0195$ and $\varepsilon = 0.0792$

Figure 2b shows the dependence of maximum pressure in the cavity on the ratio Gs/Pr. The proportionality factor is $\varepsilon = 0.0792$. It confirms the suggested linear law. Note that ε may vary with Prandtl number. In what follows, it is assumed that $0 < \varepsilon < 1$, which is expected to hold in a wide range of Prandtl numbers.

Let us now make a note on time-dependent averaged flows. In fact, this work was inspired by the need to analyze the applicability of Boussinesq approximation to describing the transient process in a short-duration microgravity experiment [13-15]. If non–uniform thermal field is established before vibrations are applied, the development of convective flow occurs during 1–2 viscous times. The calculations show that at two viscous times from the start of vibration, the dependence of Reynolds number on Gs/Pr is similar to that of Fig. 2 with slightly smaller values of the parameter γ. Therefore, one can use previously derived estimations for velocity and pressure. To estimate the time derivatives of averaged components, we note that the transfer of heat and mass in the considered transient period is mainly due to averaged motion, so the characteristic time can be estimated as length scale divided by the magnitude of averaged velocity, i.e. $L/|\overline{\mathbf{V}}|$.

3.2. Validity of Boussinesq Approximation

Let us now introduce the obtained estimations into the basic assumptions of Boussinesq approximation (see Section 1) taking into account (10) and (11). We will also employ criteria (27) and (41), which ensure the applicability of averaging approach. The latter can be rewritten as

$$\frac{\nu}{L^2\omega} \ll 1, \qquad \frac{\chi}{L^2\omega} \ll 1, \qquad \frac{\beta_T \Delta T A}{L} \ll 1. \tag{61}$$

Assumption 1 expressed by Eq. (4) leads to

$$\frac{\Delta T}{T_0} \ll 1, \qquad \frac{\rho_0 A \omega^2 L}{P_0} \ll 1. \tag{62}$$

The first inequality states that the applied temperature difference ΔT must be small in comparison with the mean temperature T_0. The second inequality requires that the pressure variations

caused by vibrational force with acceleration $A\omega^2 \cos(\omega t)$ must be small in comparison with the mean pressure P_0.

Note that here and below we present only *independent* criteria and skip the dependent ones (including those obtained by multiplying the existing criteria by $\gamma < 1$). For example, for the oscillatory temperature component, it follows from (4) that

$$\frac{\beta_T \Delta T A}{L} \frac{\Delta T}{T_0} \ll 1,$$

which is satisfied once the last inequality in (61) and the first inequality in (62) hold.

In the averaging approach, the full governing equations are separated into the equations for averaged and oscillatory fields. Therefore, we require that Assumption 2 expressed by Eq. (5) must be satisfied for averaged and oscillatory components separately (including their time derivatives). It gives the following criteria:

$$\beta_T \Delta T \ll 1, \qquad \beta_P \rho_0 \left(\frac{\omega L}{\beta_T \Delta T}\right)^2 \ll 1. \tag{63}$$

The first inequality states that the density change induced by the applied temperature difference is much smaller than the mean density ρ_0, see also equation (6). The second inequality requires that the density change due to pressure variations caused by the vibrational force must be smaller than the density change due to applied temperature difference. It should be noted that this inequality is related to the requirement that the acoustic wavelength τc (where c is the speed of sound) must be greater than the characteristic length scale L. Indeed, taking into account that $c^2 = \partial p / \partial \rho = (\beta_P \rho_0)^{-1}$, we can rewrite the above requirement as

$$\beta_P \rho_0 (\omega L)^2 \ll 1,$$

This requirement is weaker than the last one in (63) as $\beta_T \Delta T \ll 1$.

The conditions, under which Assumptions 3 and 4 of Section 1 are satisfied, are derived in two steps. At the first step, we substitute representation (10) into system (1)–(3). The otained equations are averaged over the period and the result is subtracted from the original (non–averaged) system. It provides equations for oscillatory fields. After that, we derive the criteria, under which these equations are reduced to (29)–(31). At the second step, we substitute (38), (39) into the averaged equations and calculate the related terms. Finally, the criteria, under which the resulting equations are reduced to (42)–(45), are derived.

One can consider two cases depending on whether the Reynolds number is less or greater than unity (in practice, one should check the value of $\gamma^2 \, \mathrm{Gs}/ \mathrm{Pr}$).

1. *Flows with small Reynolds number.* This case is characterized by relations (59), so we assume that

$$\mathrm{Re} \sim \left(\frac{\gamma \beta_T \Delta T A \omega L}{\nu}\right)^2 < 1. \tag{64}$$

Substituting (6) into the continuity equation (2) and deriving the equation for oscillatory fields, we find

$$-\beta_T \big(\partial_t T' + \overline{\boldsymbol{V}} \cdot \nabla T' + \boldsymbol{V}' \cdot \nabla \overline{T} + \boldsymbol{V}' \cdot \nabla T' - \overline{\boldsymbol{V}' \cdot \nabla T'} + T' \nabla \cdot \overline{\boldsymbol{V}}\big) + \tag{65}$$

$$+ \big(1 - \beta_T \overline{T} - \beta_T T'\big) \nabla \cdot \boldsymbol{V}' + \beta_T \overline{T' \nabla \cdot \boldsymbol{V}'} = 0.$$

Note that the terms $\boldsymbol{V}' \cdot \nabla T'$ and $\beta_T T' \nabla \cdot \boldsymbol{V}'$ enter into this equation together with their averaged counterparts. One should take into account that the latter cannot exceed the former by absolute value when performing the order of magnitude analysis. In what follows, all such

cases are treated in the same way. Equation (65) is reduced to (31) by requiring that all other terms are negligible in comparison with $\nabla \cdot \boldsymbol{V}'$.

The energy balance (3) provides the following relation for oscillatory fields:

$$c_p \rho_0 (1 - \beta_T \overline{T} - \beta_T T')\big(\partial_t T' + \overline{\boldsymbol{V}} \cdot \nabla T' + \boldsymbol{V}' \cdot \nabla \overline{T} + \boldsymbol{V}' \cdot \nabla T'\big) - c_p \rho_0 (1 - \beta_T \overline{T}) \overline{\boldsymbol{V}' \cdot \nabla T'} -$$

$$- c_p \rho_0 \beta_T T'\big(\partial_t \overline{T} + \overline{\boldsymbol{V}} \cdot \nabla \overline{T}\big) + c_p \rho_0 \beta_T \big(\overline{T' \partial_t T'} + \overline{T' \overline{\boldsymbol{V}} \cdot \nabla T'} + \overline{T' \boldsymbol{V}'} \cdot \nabla \overline{T} + \overline{T' \boldsymbol{V}' \cdot \nabla T'}\big) -$$

$$- \beta_T (T_0 + \overline{T} + T')\big(\partial_t P' - \rho_0 A \omega^3 \sin(\omega t)\, \boldsymbol{e} \cdot \boldsymbol{x} + \overline{\boldsymbol{V}} \cdot \nabla P' + \boldsymbol{V}' \cdot \nabla \overline{P} + \boldsymbol{V}' \cdot \nabla P' +$$

$$+ (\overline{\boldsymbol{V}} + \boldsymbol{V}') \cdot \boldsymbol{e}\, \rho_0 A \omega^2 \cos(\omega t)\big) + \beta_T (T_0 + \overline{T})\big(\overline{\boldsymbol{V}' \cdot \nabla P'} + \overline{\boldsymbol{V}' \cdot \boldsymbol{e}\, \rho_0 A \omega^2 \cos(\omega t)}\big) -$$

$$- \beta_T T'\big(\partial_t \overline{P} + \overline{\boldsymbol{V}} \cdot \overline{P}\big) + \beta_T \big(\overline{T' \partial_t P'} - \overline{T' \rho_0 A \omega^3 \sin(\omega t)}\, \boldsymbol{e} \cdot \boldsymbol{x} + \tag{66}$$

$$+ \overline{T' \overline{\boldsymbol{V}} \cdot \nabla P'} + \overline{T' \boldsymbol{V}'} \cdot \nabla \overline{P} + \overline{T' \boldsymbol{V}' \cdot \nabla P'} + \overline{T'(\overline{\boldsymbol{V}} + \boldsymbol{V}') \cdot \boldsymbol{e}\, \rho_0 A \omega^2 \cos(\omega t)}\big) +$$

$$= \kappa \nabla^2 \overline{T} + \frac{\mu}{2} \sum_{i,j=1}^{3} \left[\left(\frac{\partial V_i'}{\partial x_j} + \frac{\partial V_j'}{\partial x_i}\right)\left(2\frac{\partial \overline{V}_i}{\partial x_j} + 2\frac{\partial \overline{V}_j}{\partial x_i} + \frac{\partial V_i'}{\partial x_j} + \frac{\partial V_j'}{\partial x_i} - \frac{4}{3}\delta_{ij}\nabla \cdot \overline{\boldsymbol{V}}\right) - \right.$$

$$\left. - \overline{\left(\frac{\partial V_i'}{\partial x_j} + \frac{\partial V_j'}{\partial x_i}\right)^2} \right],$$

where we have taken into account that $\nabla \cdot \boldsymbol{V}' = 0$. This equation is reduced to (30) by comparing all terms with $\boldsymbol{V}' \cdot \nabla \overline{T}$. This term is responsible for convective transport of heat by the oscillatory velocity component. We require that all other terms except $\partial_t T'$ are negligible in comparison with it.

Finally, the momentum equation for the oscillatory components has the form

$$\rho_0 (1 - \beta_T \overline{T} - \beta_T T')\big(\partial_t \boldsymbol{V}' + (\overline{\boldsymbol{V}} \cdot \nabla)\boldsymbol{V}' + (\boldsymbol{V}' \cdot \nabla)\overline{\boldsymbol{V}} + (\boldsymbol{V}' \cdot \nabla)\boldsymbol{V}'\big) - \rho_0 (1 - \beta_T \overline{T})\overline{(\boldsymbol{V}' \cdot \nabla)\boldsymbol{V}'} -$$

$$- \rho_0 \beta_T T'\big(\partial_t \overline{\boldsymbol{V}} + (\overline{\boldsymbol{V}} \cdot \nabla)\overline{\boldsymbol{V}}\big) + \rho_0 \beta_T \big(\overline{T' \partial_t \boldsymbol{V}'} + \overline{T'(\overline{\boldsymbol{V}} \cdot \nabla)\boldsymbol{V}'} + (\overline{T' \boldsymbol{V}'} \cdot \nabla)\overline{\boldsymbol{V}} + \overline{T'(\boldsymbol{V}' \cdot \nabla)\boldsymbol{V}'}\big) =$$

$$= -\nabla P' + \mu \nabla^2 \boldsymbol{V}' - \rho_0 \beta_T (\overline{T} + T')A\omega^2 \cos(\omega t)\, \boldsymbol{e} + \overline{\rho_0 \beta_T T' A \omega^2 \cos(\omega t)}\boldsymbol{e}. \tag{67}$$

It is reduced to (29) by comparing all terms with the vibrational force term $\rho_0 \beta_T \overline{T}\, A\omega^2 \cos(\omega t)\, \boldsymbol{e}$ and using the criteria of averaging approach (61). Consideration of equations for oscillatory fields (65)–(67) provides an additional criteria

$$\frac{T_0\,(\omega L)^2}{c_p\,\Delta T^2} \ll 1. \tag{68}$$

It is derived by assuming that the term $\beta_T T_0 \rho_0 A\omega^3 \sin(\omega t)\, \boldsymbol{e} \cdot \boldsymbol{x}$ is small in comparison with $\boldsymbol{V}' \cdot \nabla \overline{T}$ in equation (66).

One can show that inequality (68) can be derived from the previously obtained criteria. Consider the following thermodynamic relation [24]

$$\left(\frac{\partial V}{\partial P}\right)_S = \left(\frac{\partial V}{\partial P}\right)_T + \frac{T_0}{c_p}\left(\frac{\partial V}{\partial T}\right)_P^2,$$

where $V = \rho^{-1}$ is the specific volume and S is the specific enthropy. Let us rewrite it in the form

$$\frac{T_0}{c_p} - \frac{\beta_P \rho_0}{\beta_T^2} = \left(\frac{\rho_0}{\beta_T}\right)^2 \left(\frac{\partial V}{\partial P}\right)_S.$$

The left–hand side of this relation is negative since $(\partial V/\partial P)_S < 0$ [24]. Combining this result with the second criterium in (63), one can see that if the latter is satisfied, then (68) also holds.

Let us now substitute decomposition (10) with the expressions for oscillatory components (38) and (39) into the governing equations (1)–(3) and average them over the period. The averaged continuity equation is given by

$$-\beta_T(\partial_t \overline{T} + \overline{\mathbf{V}} \cdot \nabla \overline{T}) + (1 - \beta_T \overline{T})\nabla \cdot \overline{\mathbf{V}} = 0.$$

We require that all terms except $\nabla \cdot \overline{\mathbf{V}}$ are negligible here, so the continuity equation is reduced to (44). Taking it into account, the averaged energy equations is written as

$$c_p \rho_0 (1 - \beta_T \overline{T})(\partial_t \overline{T} + \overline{\mathbf{V}} \cdot \nabla \overline{T}) - \beta_T(T_0 + \overline{T})(\partial_t \overline{P} + \overline{\mathbf{V}} \cdot \overline{P}) - \tag{69}$$

$$-\rho_0 \frac{(\beta_T A)^2}{2}((\overline{T}\boldsymbol{e} - \nabla\Phi) \cdot \nabla \overline{T})\,\overline{\mathbf{V}} \cdot \left(c_p \beta_T \nabla((\overline{T}\boldsymbol{e} - \nabla\Phi) \cdot \nabla \overline{T}) + \beta_T \omega^2 \nabla \Phi - \omega^2 \boldsymbol{e}\right) =$$

$$= \kappa \nabla^2 \overline{T} + \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial \overline{V}_i}{\partial x_j} + \frac{\partial \overline{V}_j}{\partial x_i}\right)^2 + \frac{\mu(\beta_T A \omega)^2}{4} \sum_{i,j=1}^{3} \left(\frac{\partial W_i}{\partial x_j} + \frac{\partial W_j}{\partial x_i}\right)^2,$$

where $\mathbf{W} = (W_1, W_2, W_3)$, see (32). According to Assumption 4 in Section 1, we require that the contribution of viscous energy dissipation and pressure variations to the energy balance is negligible (as well as the contribution of the last three terms in the left–hand side of Eq. (69)). These terms are compared with the term $\overline{\mathbf{V}} \cdot \nabla \overline{T}$, which describes convective heat transfer. As a result, the energy equation is reduced to (44). It should be noted that the last term in (69) results from time averaging of the term $\frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial V_i'}{\partial X_j} + \frac{\partial V_j'}{\partial X_i}\right)^2$ in the oscillatory part of governing equations (see (66)). Therefore, the viscous energy dissipation due to oscillatory motion contributes to the time–averaged energy balance. The requirement that this contribution is negligible in comparison with convective heat transfer described by the term $\overline{\mathbf{V}} \cdot \nabla \overline{T}$ provides an additional criteria

$$\frac{1}{\mathrm{Re}} \times \left(\frac{\beta_T \Delta T A}{L}\right)^2 \times \frac{T_0(\omega L)^2}{c_p \Delta T^2} \times \frac{\Delta T}{T_0} \ll 1, \tag{70}$$

where the Reynolds number is determined by (64). Note that due to (61), (62), (68), the product of the last three factors in the left–hand side of (70) has the order of 10^{-4} or lower. It follows that viscous dissipation can be important only for mean flows with small Reynolds number ($\mathrm{Re} < 10^{-3}$). The above criterium can be rewritten as

$$\frac{\nu^2}{\gamma^2 L^2 \, c_p \, \Delta T} \ll 1. \tag{71}$$

Note that if we assume that a stronger inequality than (68)

$$\frac{T_0 \, (\omega L)^2}{\gamma^2 c_p \, \Delta T^2} \ll 1 \tag{72}$$

is satisfied, then criterium (71) will hold automatically. Indeed, the left–hand side of (71) is obtained by multiplying that of (72) by $\nu^2/L^4\omega^2$ and $\Delta T/T_0$, which are small due to (61) and (62), respectively. It should be noted that in contrast to (68), inequality (72) is not automatically satisfied once the second criterium in (63) holds.

Finally, the averaged momentum equation is written as

$$\rho_0(1 - \beta_T \overline{T})(\partial_t \overline{\mathbf{V}} + (\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}}) =$$

Table 1. Physical parameters and characteristics of the system (isopropanol in a cubic cell)

T_0	P_0	ρ_0	β_T	β_P	ν	χ	c_p	L	ΔT
K	Pa	kg/m^3	10^{-3} K^{-1}	10^{-9} Pa^{-1}	10^{-6} m^2/s	10^{-7} m^2/s	J/kg K	m	K
313	101325	769	1.095	1.02	1.730	0.623	2735	0.005	20

$$= -\nabla \overline{P} + \mu \nabla^2 \overline{\mathbf{V}} + \frac{\rho_0 (\beta_T A \omega)^2}{2} \big((T\boldsymbol{e} - \nabla \Phi) \cdot \nabla \big) \big(\nabla \Phi + \beta_T \overline{T}(\overline{T}\boldsymbol{e} - \nabla \Phi) \big).$$

It is reduced to (42) on the basis of already derived criteria.

2. Flows with large Reynolds number. Such flows are described by relations (60), so we suppose that

$$\text{Re} \sim \frac{\gamma \beta_T \Delta T A \omega L}{\nu} \geqslant 1. \tag{73}$$

This case is treated similar and provides the same criteria. However, relation (70) with the Reynolds number given by (73) is satisfied automatically as Re $\geqslant$ 1.

3.3. Example: Isopropanol in a Cubic Cavity

In this section, we consider an example of a real physical system. Experimental studies of thermovibrational convection in microgravity conditions have been recently performed in [14]. In this work, vibrations were applied to a cubic cell filled with isopropanol at atmospheric pressure. The physical parameters and characteristics of this system are given in Table 1. In the experiments, the temperature difference was chosen as $\Delta T = 20$ K. It ensures that the first inequalities in (62) and (63) as well as inequality (71) are satisfied.

Applying criteria (61)–(63), (71) to the present system, we obtain the following restrictions on the frequency and amplitude of vibration: $0.011 \ll f \ll 787$ Hz, $A \ll 0.228$ m, $A(2\pi f)^2 \ll 26352$ m^2/s. Here the lower and upper frequency limits follow from the first inequality in (61) and second inequality in (63), respectively. The amplitude limitation results from the last inequality in (61), while the vibrational acceleration is limited by the second inequality in (62). The above requirements ensure the applicability of Boussinesq approximation and averaging approach to the description of the considered physical system.

Conclusion

In this work, we have analyzed the applicability of Boussinesq approximation to the problems of thermovibrational convection in closed volumes. The study is focused on the limit of high frequency and small amplitude, where the averaging technique can be effectively applied. A rigorous derivation of averaged equations for thermovibrational convection is presented including the details, which are not discussed in the existing literature. Using the order of magnitude analysis, we have estimated the magnitudes of oscillatory and averaged flows fields. Two cases, which correspond to averaged flows with small and large Reynolds numbers Re, were distinguished. It was found that Re $\sim \gamma^2 \text{Gs}/\text{Pr}$ for Re < 1 and Re $\sim \gamma(\text{Gs}/\text{Pr})^{1/2}$ for Re $\geqslant 1$, where Gs and Pr are the Gershuni and Prandtl numbers, respectively, while $\gamma < 1$ represents the ratio of averaged velocity magnitude to the oscillatory velocity magnitude. The obtained estimations correlated with numerical modelling provide new essential information about the intensity of averaged flows in a wide range of vibration stimuli. These estimations are used to derive the criteria for the validity of Boussinesq approximation. The basic assumptions of Boussinesq approach are applied

to the momentum, continuity, and energy equations for a compressible, viscous heat–conducting fluid. The contribution of viscous energy dissipation and pressure work to the energy balance is taken into account. The order of magnitude analysis provides a number of dimensionless parameters, the smallness of which guarantees the validity of Boussinesq approximation. It is found that if the density variation with pressure is much smaller than that with temperature, then the contribution of pressure work to the energy balance can be neglected. The contribution of viscous dissipation due to oscillatory motion to the averaged energy balance can be important only for mean flows with small Reynolds number. An example of a concrete physical system is considered. The results of this work provide important and useful information for further analytical and numerical studies of vibrational phenomena in fluids.

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О приближении Буссинеска в задачах конвекции, вызванной высокочастотными вибрациями

Илья И. Рыжков
Юрий А. Гапоненко

Изучена применимость приближения Буссинеска к задачам термовибрационной конвекции в замкнутых объемах. Рассматриваются вибрации высокой частоты и малой амплитуды, что позволяет использовать метод осреднения. Получены оценки пульсационных и осредненных компонент течения. Показано, что число Рейнольдса для осредненного движения пропорционально отношению числа Гершуни к числу Прандтля в области малых чисел Рейнольдса и корню квадратного из этого отношения в области больших чисел Рейнольдса. Эти результаты дают важную информацию о характеристиках осредненных течений в зависимости от интенсивности вибрации. Принимая во внимание полученные оценки, основные предположения приближения Буссинеска применены к уравнениям сохранения массы, импульса и энергии для сжимаемой вязкой теплопроводной жидкости (газа). Учитывается вклад вязкой диссипации и работы сил давления в уравнение сохранения энергии. Анализ порядка величин различных членов в уравнениях движения приводит к набору безразмерных параметров, малость которых гарантирует справедливость приближения Буссинеска.

Ключевые слова: приближение Буссинеска, вибрационная конвекция, осредненное движение.