DOI: 10.17516/1997-1397-2020-13-2-231-241

УДК 519.716

E-closed Sets of Hyperfunctions on Two-Element Set

Vladimir I. Panteleyev*

Leonid V. Riabets†

Irkutsk State University
Irkutsk, Russian Federation

Received 09.11.2019, received in revised form 06.01.2020, accepted 13.02.2020

Abstract. Hyperfunctions are functions that are defined on a finite set and return all non-empty subsets of the considered set as their values. This paper deals with the classification of hyperfunctions on a two-element set. We consider the composition and the closure operator with the equality predicate branching (*E*-operator). *E*-closed sets of hyperfunctions are sets that are obtained using the operations of adding dummy variables, identifying variables, composition, and *E*-operator. It is shown that the considered classification leads to a finite set of closed classes. The paper presents all 78 *E*-closed classes of hyperfunctions, among which there are 28 pairs of dual classes and 22 self-dual classes. The inclusion diagram of the *E*-closed classes is constructed, and for each class, its generating system is obtained.

Keywords: closure, equality predicate, hyperfunction, closed set, composition.

Citation: V.I.Panteleyev, L.V.Riabets, *E*-closed Sets of Hyperfunctions on Two-Element Set, J. Sib. Fed. Univ. Math. Phys., 2020, 13(2), 231-241. DOI: 10.17516/1997-1397-2020-13-2-231-241.

Systems with generalization of k -valued logic functions have been studied for a long time along with classical functional systems over a set of k -valued functions ($k \geq 2$). Such systems based on partial functions, multifunctions, hyperfunctions. These functions defined on a finite set A and taking values in the set of subsets of A . Usually such systems closed with respect to the composition operator (see [1–7]).

The composition operator leads to a countable or continuous classification; therefore, closure operators that generate finite classifications of functions are of interest. Such operators, in particular, include the parametric and positive closure operators [8], the operator with the equality predicate branching (*E*-operator) [9]. An investigation of *E*-operator on the set of Boolean functions, partial Boolean functions and on the set of functions of k -valued logic can be found in [9–11]. All *E*-closed classes for the set of partial Boolean functions were obtained in [12]. The complete structure of closed classes for parametric and positive closure operators for hyperfunctions on two-element set was obtained in [13, 14]. The completeness criterion for the *E*-operator on the set of hyperfunctions of rank two was proved in [15].

The aim of this paper is to describe all *E*-closed classes of hyperfunctions on two-element set.

Introduction

Let $E_2 = \{0, 1\}$ and $\mathcal{P}(E_2)$ be the power set of E_2 . An n -ary hyperfunction f on E_2 is a mapping

$$f : E_2^n \rightarrow \mathcal{P}(E_2) \setminus \{\emptyset\}.$$

*vl.panteleyev@gmail.com <https://orcid.org/0000-0003-4766-486X>

†l.riabets@gmail.com <https://orcid.org/0000-0003-4047-9573>

© Siberian Federal University. All rights reserved

We will write P_2^- for $\mathcal{P}(E_2) \setminus \{\emptyset\}$. Let $H_{2,n} = (P_2^-)^{E_2^n}$ be the set of all n -ary hyperfunctions on A , $n \geq 1$, and $H_2 = \bigcup_n H_{2,n}$ be the set of all finitary hyperfunctions on E_2 .

An i -th n -ary projection (selector hyperfunction) on E_2 , $1 \leq i \leq n$, is the n -ary hyperfunction $e_i^n \in H_{2,n}$ defined by $e_i^n(x_1, \dots, x_n) = \{x_i\}$.

In what follows, we will not distinguish between a set of one element and an element of this set. For the set E_2 , we will use the notation "–" (dash).

The n -variable hyperfunction f will be represented as a vector $(\tau_{\tilde{0}}, \dots, \tau_{\tilde{1}})$, where $\tilde{0}, \dots, \tilde{1}$ are binary representations of numbers $0, \dots, 2^n - 1$ and $\tau_{\tilde{\sigma}}$ equals to $f(\tilde{\sigma})$. Such vectors have the form $(f(0)f(1))$ for unary hyperfunctions and $(f(0,0)f(0,1)f(1,0)f(1,1))$ for binary hyperfunctions.

Let $f \in H_{2,n}$ and $f_1, \dots, f_n \in H_{2,m}$ for positive integers n and m . The composition of hyperfunctions f and $f_1, \dots, f_n$ is the n -ary hyperfunction $f(f_1, \dots, f_n)$ defined by

$$f(f_1, \dots, f_n)(\alpha_1, \dots, \alpha_m) = \bigcup_{\beta_i \in f_i(\alpha_1, \dots, \alpha_m)} f(\beta_1, \dots, \beta_n),$$

where $(\alpha_1, \dots, \alpha_m) \in E_2^m$.

We say that the hyperfunction $g(x_1, \dots, x_n)$ is obtained from the functions $f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n)$ using the operator with the equality predicate branching (E -operator) if for some $i, j \in \{1, \dots, n\}$ the following relation holds:

$$g(x_1, \dots, x_n) = \begin{cases} f_1(x_1, \dots, x_n) & \text{if } x_i = x_j, \\ f_2(x_1, \dots, x_n) & \text{otherwise.} \end{cases}$$

The set of all hyperfunctions H_2 that can be obtained from the set $Q \subseteq H_2$ using the operations of adding dummy variables, identifying variables, composition and equality predicate branching is called E -closure of set Q .

A set of hyperfunctions that coincides with its closure is called an E -closed class. We say that the set $R \subseteq Q$ E -generates an E -closed class Q if E -closure of the set R coincides with the class Q . Therefore R is an E -complete in Q . Following [10] the E -closure of R is denoted by $[R]_E$. By $Q(n)$ denote a set of all n -variable hyperfunctions from Q .

The hyperfunction g is called dual to the hyperfunction f , if $g(x_1, \dots, x_n) = \bar{f}(\bar{x}_1, \dots, \bar{x}_n)$. A class that includes all hyperfunctions dual to hyperfunctions of class Q , is called dual to the class Q and denoted $\bar{Q}$. The class Q will be called self-dual if $Q = \bar{Q}$.

The original hyperfunction and the hyperfunctions obtained from it by identifying variables or adding dummy variables will be denoted by the same symbols, if this does not cause confusion.

1. The extended operator with the equality predicate branching

The operator with the equality predicate branching allows us to reduce the E -closure of H_2 to the E -closure of sets of 2-variable hyperfunctions.

Proposition 1.1. *Any E -closed $Q \subseteq H_2$ is E -generated by the set of all its hyperfunctions depending on at most two variables.*

Proof. The proof is similar to the proof of the corresponding statement in [10].

Proposition 1.1 shows that there are finite number of E -closed classes in H_2 . Moreover, for any two E -closed classes Q_1 and Q_2 , $Q_1 \subseteq Q_2$ is equivalent to $Q_1(2) \subseteq Q_2(2)$. Thus the problem

of describing all *E*-closed classes in H_2 is reduced to the construction of all *E*-closed sets of 2-variable hyperfunctions. Before presenting an algorithm for constructing such sets let us pay attention to the following fact (the similar fact is mentioned in [12]).

Consider hyperfunctions $f_1, \dots, f_k$ depending on no more than n variables. If we try to obtain n -variable hyperfunction f using $f_1, \dots, f_k$, we may need to use intermediate hyperfunctions that depend on more than n variables.

As an example, let $f_1(x_1, x_2) = (-101)$, $f_2(x_1, x_2) = (-011)$, and $f_3(x_1, x_2) = (-1 - 1)$. Applying composition operator and *E*-operator to f_1, f_2, f_3 without using hyperfunctions of a larger number of variables allows us to obtain only f_1, f_2 , or f_3 .

Let $g(x_1, x_2, x_3)$ be a hyperfunction such that

$$g(x_1, x_2, x_3) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 = x_3, \\ f_2(x_1, x_2) & \text{otherwise.} \end{cases}$$

Now we can obtain $f(x_1, x_2) = (-0 - 1)$:

$$f(x_1, x_2) = g(x_1, x_2, f_3(x_2, x_2)).$$

Thus to work with hyperfunctions of no more than two variables, it is necessary to make more precise the definitions of the used operators.

Let f, g_1, g_2, h_1 , and h_2 be 2-variable hyperfunctions. We say that the hyperfunction f is obtained from the functions g_1, g_2, h_1, h_2 using an extended operator with the equality predicate branching (*Ex*-operator) if for any binary set $(\alpha_1, \alpha_2) \in E_2^2$ the following relation holds:

- if $h_1(\alpha_1, \alpha_2), h_2(\alpha_1, \alpha_2) \in E_2$, then

$$f(\alpha_1, \alpha_2) = \begin{cases} g_1(\alpha_1, \alpha_2) & \text{if } h_1(\alpha_1, \alpha_2) = h_2(\alpha_1, \alpha_2), \\ g_2(\alpha_1, \alpha_2) & \text{otherwise;} \end{cases}$$

- if $h_1(\alpha_1, \alpha_2) = -$ or $h_2(\alpha_1, \alpha_2) = -$, then

$$f(\alpha_1, \alpha_2) = g_1(\alpha_1, \alpha_2) \cup g_2(\alpha_1, \alpha_2).$$

For brevity, we use the notation:

$$f(x_1, x_2) = Ex(g_1(x_1, x_2), g_2(x_1, x_2), h_1(x_1, x_2), h_2(x_1, x_2)).$$

Further, we will consider the composition operator (restricted composition) only in the following form:

$$g_1(h_1(x_1, x_2), h_2(x_1, x_2)).$$

For the above-defined operators hyperfunctions h_1 and h_2 can be selector hyperfunctions. The closure of the set Q obtained with respect to the extended operator with the equality predicate branching, restricted composition, operation of adding dummy variables, and identifying variables will be denoted by $[Q]_{Ex}$.

2. *Ex*-closed classes of H_2

The definition of *Ex*-closure allows us to formulate an algorithm for constructing *Ex*-closed classes of hyperfunctions.

The algorithm constructs *Ex*-generated sets of hyperfunctions. Each 2-variable hyperfunction will be associated with its number from 0 to 80. At each iteration, a sequence of restricted compositions and a sequence of extended operations with the equality predicate branching are applied.

The algorithm builds one-element and two-element *Ex*-generated sets separately. Computer calculations showed that there are no three-element *Ex*-generated sets. We describe the steps of the algorithm in the form of pseudo-code.

```
function get_hyperfunction_classes() {
  vars
    F: collection<function>;
    Q: collection<class>;
    A: class;
  for each f in  $H_2$  do {
    A = new class;
    A  $\leftarrow$  f;
    while has_new(A) do {
      A  $\leftarrow$  composition(A);
      A  $\leftarrow$  Ex(A);
    }
    if is_new(Q, A) then{
      Q  $\leftarrow$  A;
      F  $\leftarrow$  f;
    }
  }
  while has_new(Q) do {
    for each B in Q do {
      for each f in F do {
        B  $\leftarrow$  f;
        while has_new(B) do {
          B  $\leftarrow$  composition(B);
          B  $\leftarrow$  Ex(B);
        }
        if is_new(Q, B) then{
          Q  $\leftarrow$  B;
        }
      }
    }
  }
  return Q;
}
```

The algorithm was implemented in Java. It was found that there are precisely 78 *Ex*-closed classes of H_2 . Among them, there are 56 classes that are divided into pairs of pairwise dual

classes and, 22 classes are self-dual.

Now we list the known *E*-closed classes of H_2 obtained in [15]:

$$T_0^{0-} = \{f(x_1, \dots, x_n) \mid f(0, \dots, 0) \in \{0, -\}\};$$

$$T_1^{1-} = \{f(x_1, \dots, x_n) \mid f(1, \dots, 1) \in \{1, -\}\};$$

$$S^- = \{f(x_1, \dots, x_n) \mid (f(\alpha_1, \dots, \alpha_n), f(\bar{\alpha}_1, \dots, \bar{\alpha}_n)) \notin \{(0, 0), (1, 1)\}, (\alpha_1, \dots, \alpha_n) \in E_2^n\};$$

$$O_2 = \{f(x_1, \dots, x_n) \mid f(\alpha_1, \dots, \alpha_n) \in \{0, 1\}, (\alpha_1, \dots, \alpha_n) \in E_2^n\}.$$

According to the main theorem in [15] these classes are *E*-precomplete in H_2 .

In [9] was shown that the well-known classes of Boolean functions T_0 , T_1 , S , T_{01} , S_{01} , C_0 , and C_1 are *E*-closed.

The self-dual classes $U_1, \dots, U_{16}$, H_2 , S^- , O_2 , S , S_{01} , T_{01} and non-dual classes $V_1, \dots, V_{25}$, T_0^{0-} , T_0 , C_0 (one from each pair) are presented in the form of an inclusion diagram in Fig 1.

Additional information on *Ex*-closed classes is presented in Tab. 1. The first column shows the name of *Ex*-closed class, the second column shows the number of 2-variable hyperfunctions in the class, and the third column shows the class generating system. One representative class from the pair of pairwise dual classes is presented in Tab. 1.

Fig. 1. The diagram of inclusions for *Ex*-closed classes of H_2

3. *E*-closed classes of H_2

The result obtained in the previous section allows us to formulate an upper bound for *E*-closed classes of hyperfunctions.

Theorem 3.1. *For any set $Q \subseteq H_2$, it follows that $[Q]_{Ex} \subseteq [Q]_E$.*

Proof. We show that each hyperfunction obtained using extended operators with the equality predicate branching and restricted composition can be represented by a formula using the *E*-closure operators.

Table 1. The generating sets for *Ex*-closed classes

1	2	3	1	2	3
H_2	81	(0000), (1 – 00)	V_{14}	9	(011–)
T_0^{0-}	54	(0000), (–100)	U_9	9	(1 – –0)
S^-	49	(1 – 00)	V_{15}	8	(000–), (–00–)
U_1	36	(0001), (–00–)	V_{16}	8	(0 – 00)
V_1	35	(–100), (0101)	T_0	8	(0100)
U_2	28	(0101), (– – 0–)	V_{17}	8	(–000)
U_3	27	(–001), (000–)	U_{10}	7	(–10–), (– – 0–)
V_2	27	(0 – 10)	U_{11}	7	(0 – 01)
V_3	27	(–110)	V_{18}	7	(010–)
U_4	25	(– – 00), (1 – 0–)	V_{19}	6	(0 – 0–), (– – 0–)
U_5	21	(–101), (010–)	U_{12}	6	(0101), (–10–)
V_4	21	(–100)	V_{20}	6	(– – 00)
V_5	20	(– – 00), (0 – 1–)	U_{13}	5	(– – 0–), (– – 1–)
V_6	18	(000–), (–10–)	V_{21}	5	(0 – 1–)
V_7	18	(0001), (000–)	V_{22}	4	(–00–)
V_8	16	(0000), (–000)	V_{23}	4	(000–)
O_2	16	(1000)	T_{01}	4	(0001)
U_6	15	(– – 01), (0 – 0–)	S	4	(1100)
V_9	15	(– – 10)	V_{24}	3	(– – 0–)
V_{10}	14	(010–), (–10–)	U_{14}	3	(–10–)
V_{11}	14	(0101), (010–)	U_{15}	3	(0 – –1)
V_{12}	10	(0 – 0–), (– – 1–)	V_{25}	3	(0 – 0–)
U_7	9	(–00–), (–10–)	S_{01}	2	(0101)
U_8	9	(0001), (0 – 01)	U_{16}	1	(– – – –)
V_{13}	9	(– – 00), (0 – 0–)	C_0	1	(0000)

Consider

$$f(x, y, z, t) = \begin{cases} g_1(x, y) & \text{if } z = t, \\ g_2(x, y) & \text{otherwise.} \end{cases}$$

Let

$$U(x, y) = f(x, y, h_1(x, y), h_2(x, y)).$$

Now we obtain its possible values on some binary set (α_1, α_2) for various hyperfunctions h_1 and h_2 . Let $h_1(\alpha_1, \alpha_2) = \tau_1$ and $h_2(\alpha_1, \alpha_2) = \tau_2$. Consider all possible values for τ_1 and τ_2 .

- Let $\tau_1 \in E_2$ and $\tau_2 \in E_2$. Then

$$U(\alpha_1, \alpha_2) = f(\alpha_1, \alpha_2, \tau_1, \tau_2) = \begin{cases} g_1(\alpha_1, \alpha_2) & \text{if } \tau_1 = \tau_2, \\ g_2(\alpha_1, \alpha_2) & \text{otherwise.} \end{cases}$$

- Let $\tau_1 = -$ и $\tau_2 \in E_2$. Substitute these values

$$U(\alpha_1, \alpha_2) = f(\alpha_1, \alpha_2, -, \tau_2) = f(\alpha_1, \alpha_2, 0, \tau_2) \cup f(\alpha_1, \alpha_2, 1, \tau_2) = g_1(\alpha_1, \alpha_2) \cup g_2(\alpha_1, \alpha_2).$$

The case $\tau_1 \in E_2$ and $\tau_2 = -$ is similar to the previous one.

- Let $\tau_1 = -$ и $\tau_2 = -$. Then

$$\begin{aligned} U(\alpha_1, \alpha_2) &= f(\alpha_1, \alpha_2, -, -) = \\ &= f(\alpha_1, \alpha_2, 0, 0) \cup f(\alpha_1, \alpha_2, 0, 1) \cup f(\alpha_1, \alpha_2, 1, 0) \cup f(\alpha_1, \alpha_2, 1, 1) = \\ &= g_1(\alpha_1, \alpha_2) \cup g_2(\alpha_1, \alpha_2). \end{aligned}$$

Therefore the values of $U(\alpha_1, \alpha_2)$ coincide with the values of an extended operator with the equality predicate branching based on the functions g_1 , g_2 , h_1 , and h_2 . This proves that $[Q]_E \supseteq [Q]_{Ex}$. $\square$

Corollary 1. *The number of E-closed classes is at most 78.*

To obtain a lower bound for the number of E-closed classes, we consider the following sets of hyperfunctions:

$$K_1 = T_0^{0-} = \{f(x_1, \dots, x_n) \mid f(0, \dots, 0) \in \{0, -\}\};$$

$$K_2 = \{f(x_1, \dots, x_n) \mid f(0, \dots, 0) = 0\};$$

$$K_3 = \{f(x_1, \dots, x_n) \mid f(0, \dots, 0) = -\};$$

$$K_4 = T_1^{1-} = \{f(x_1, \dots, x_n) \mid f(1, \dots, 1) \in \{1, -\}\};$$

$$K_5 = \{f(x_1, \dots, x_n) \mid f(1, \dots, 1) = 1\};$$

$$K_6 = \{f(x_1, \dots, x_n) \mid f(1, \dots, 1) = -\};$$

$$K_7 = O_2 \text{ is a set of Boolean functions};$$

$$K_8 = \{f(x_1, \dots, x_n) \mid f(\alpha_1, \dots, \alpha_n) \in \{0, -\}, (\alpha_1, \dots, \alpha_n) \in E_2^n\};$$

$$K_9 = \{f(x_1, \dots, x_n) \mid f(\alpha_1, \dots, \alpha_n) \in \{1, -\}, (\alpha_1, \dots, \alpha_n) \in E_2^n\};$$

$$K_{10} = S^- \text{ is a set of self-dual hyperfunctions};$$

K_{11} is a set of hyperfunctions such that for $R = \{(01), (10), (--)\}$, for any n , and for any $(\alpha_{11}, \alpha_{12}), \dots, (\alpha_{n1}, \alpha_{n2}) \in R$, it follows that $(f(\alpha_{11}, \dots, \alpha_{n1}), f(\alpha_{12}, \dots, \alpha_{n2})) \in R$;

K_{12} is a set of hyperfunctions that take values $(0-), (-0), (1-), (-1)$, or $(--)$ on any pair of opposite binary sets;

$$K_{13} = \{f \mid f \in T_0^{0-} \cap T_1^{1-} \text{ and } - \in \{f(0, \dots, 0), f(1, \dots, 1)\}\}.$$

Proposition 3.1. *The sets $K_1, \dots, K_{13}$ are pairwise coinciding.*

It is evident that sets K_1 – K_{10} are E-closed classes.

Lemma 3.1. *The set K_{11} is an E-closed class.*

Proof. Let hyperfunction f be obtained by a composition of hyperfunctions $g, g_1, \dots, g_m \in K_{11}$. Suppose that $f \notin K_{11}$. Thus on the sets of R , it takes the values $(0-), (-0), (1-), (-1), (00)$, or (11) . At the same time, R contains only three sets. It can be assumed that f depends on three variables.

Now consider the case for the pair $(0-)$. Let $f(01-) = 0$ and $f(10-) = -$. From the second equality it follows that there is a binary set (10α) such that $f(10\alpha) = 0$ or $f(10\alpha) = -$. By definition, if $f(01-) = 0$, then $f(01\bar{\alpha}) = 0$. Thus on binary sets $f(01\bar{\alpha}) = 0$ and $f(10\alpha) = 0$ or $f(01\bar{\alpha}) = 0$ and $f(10\alpha) = -$. Note also that $(00) \notin R$ and $(0-) \notin R$. At the same time $(\bar{\alpha}\alpha) \in R$.

On the other hand, consider the composition $g(g_1(x_1, x_2, x_3), \dots, g_m(x_1, x_2, x_3))$ on sets $(01\bar{\alpha})$ and (10α) . Since $g, g_1, \dots, g_m \in K_{11}$, we have $(g_1(01\bar{\alpha})g_1(10\alpha)), \dots, (g_m(01\bar{\alpha})g_m(10\alpha)) \in R$. It follows that

$$(g(g_1(01\bar{\alpha}), \dots, g_m(01\bar{\alpha})), g(g_1(10\alpha), \dots, g_m(10\alpha))) \in R.$$

We get a contradiction. The remaining pairs $(-0), (1-), (-1), (00)$, and (11) are verified similarly.

Consider the operator with the equality predicate branching. Let

$$f(x_1, \dots, x_n) = \begin{cases} g_1(x_1, \dots, x_n) & \text{if } x_i = x_j, \\ g_2(x_1, \dots, x_n) & \text{otherwise,} \end{cases}$$

where $g_1, g_2 \in K_{11}$.

Consider the value of $f(x_1, \dots, x_n)$ on sets of R . By definition, if $\alpha_s \in \{0, 1, -\}$, $s = \overline{1, n}$, then

$$f(\alpha_1, \dots, \alpha_n) = \bigcup_{\beta_s \in \alpha_s} f(\beta_1, \dots, \beta_n) = \bigcup_{\beta_i = \beta_j} g_1(\beta_1, \dots, \beta_n) \bigcup_{\beta_i \neq \beta_j} g_2(\beta_1, \dots, \beta_n).$$

Since $\bar{\beta}_s \in \bar{\alpha}_s \Leftrightarrow \beta_s \in \alpha_s$ and $\beta_s = \beta_t \Leftrightarrow \bar{\beta}_s = \bar{\beta}_t$, $t \in \{1, \dots, n\}$, we have

$$f(\bar{\alpha}_1, \dots, \bar{\alpha}_n) = \bigcup_{\gamma_s \in \bar{\alpha}_s} f(\gamma_1, \dots, \gamma_n) = \bigcup_{\substack{\beta_i = \beta_j \\ \beta_s \in \alpha_s}} g_1(\bar{\beta}_1, \dots, \bar{\beta}_n) \bigcup_{\substack{\beta_i \neq \beta_j \\ \beta_s \in \alpha_s}} g_2(\bar{\beta}_1, \dots, \bar{\beta}_n).$$

It is easily shown that the set R is closed with respect to the operation of joining sets. In other words, if $(\alpha_1, \alpha_2) \in R$ and $(\beta_1, \beta_2) \in R$, then $(\alpha_1 \cup \beta_1, \alpha_2 \cup \beta_2) \in R$. This completes the proof. $\square$

Lemma 3.2. *The set K_{12} is an *E*-closed class.*

Proof. The validity of the statement for the operator with the equality predicate branching is obvious. We can use the assumption of contradiction to prove that K_{12} is closed with respect to composition. $\square$

It can be shown in the usual way that K_{13} is an *E*-closed class.

Theorem 3.2. *Ex-closed classes are *E*-closed.*

Proof. Let us prove that the described above *Ex*-closed classes are different with respect to the *E*-closure. For each *Ex*-closed class K , we construct the vector $v_K = (\gamma_K^1, \dots, \gamma_K^{13})$, which indicates that the class K is a subset of $K_1 - K_{13}$:

$$\gamma_K^i = \begin{cases} 1 & \text{if } K \subseteq K_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, if $\gamma_K^j = 1$ and $\gamma_{K'}^j = 0$, then $[K]_E \neq [K']_E$.

For the convenience of comparison, we divide the set of all vectors v_K into four groups with respect to hyperfunctions belonging to the precomplete classes T_0^{0-} (*E*-closed class K_1) and T_1^{1-} (*E*-closed class K_4). By $K_1 \bar{K}_4$ we denote the set of hyperfunctions belonging to K_1 and not belonging to K_4 . The other sets we denote as $K_1 K_4$, $\bar{K}_1 K_4$, $\bar{K}_1 \bar{K}_4$. Note also that *Ex*-closed classes belonging to different groups are different with respect to the *E*-closure. In the tables below, we replace the character 0 with an empty cell in each of v_K .

Group $\bar{K}_1 \bar{K}_4$. The classes of this group are distinguished by the sets $K_7, K_{10}, K_{11}, K_{12}$ (see Tab. 2).

Group $\bar{K}_1 K_4$ and group $K_1 \bar{K}_4$. The sets in these classes are dual; therefore, if there exists a hyperfunction $f \in K$ from one group, then there exists a hyperfunction $f^* \in K^*$ from another group. Thus, if the hyperfunctions of the first group are distinguished by the sets $M_1, \dots, M_t$, then dual functions of the second group are distinguished by the dual sets $M_1^*, \dots, M_t^*$.

The classes of $K_1 \bar{K}_4$ are distinguished by the sets $K_2, K_3, K_7, K_8, K_{10}, K_{12}$ (see Tab. 3).

Table 2. Classes of $\overline{K}_1\overline{K}_4$

N		B	K_7	K_{10}	K_{11}	K_{12}
1	S	(1100)	1	1	1	
2	U_9	(1 - -0)		1	1	
3	O_2	(1000)	1			

N		B	K_7	K_{10}	K_{11}	K_{12}
4	U_4	(- - 00), (1 - 0-)		1		1
5	S^-	(1 - 00)		1		
6	H_2	(0000), (1 - 00)				

Table 3. Classes of $K_1\overline{K}_4$

N		B	K_2	K_3	K_7	K_8	K_{10}	K_{12}
7	C_0	(0000)	1		1	1		
8	V_{20}	(- - 00)		1		1	1	1
9	V_{17}	(-000)		1		1		
10	V_{16}	(0 - 00)	1			1		
11	V_{13}	(- - 00), (0 - 0-)				1	1	1
12	V_8	(0000), (-000)				1		
13	T_0^{0-}	(0000), (-100)						

N		B	K_2	K_3	K_7	K_8	K_{10}	K_{12}
14	V_5	(- - 00), (0 - 1-)					1	1
15	V_4	(-100)		1			1	
16	V_3	(-110)		1				
17	V_2	(0 - 10)	1					
18	V_1	(-100), (0101)					1	
19	V_9	(- - 10)		1			1	1
20	T_0	(0100)	1		1			

The group $\overline{K}_1\mathbf{K}_4$ contains 14 sets dual to the sets of $\mathbf{K}_1\overline{K}_4$: $C_1, V_{20}^*, V_{17}^*, V_{16}^*, V_{13}^*, V_8^*, T_1^{1-}, V_5^*, V_4^*, V_3^*, V_2^*, V_1^*, V_9^*, T_1$. These classes are distinguished by the sets $K_5, K_6, K_7, K_9, K_{10}, K_{12}$.

Group $\mathbf{K}_1\mathbf{K}_4$. We divide $K_1 \cap K_4$ into two subsets with respect to belonging to the E -precomplete class of self-dual hyperfunctions S^- (E -closed class K_{10}). The sets are presented in Tabs. 4 and 5. The enumeration of sets is performed, taking into account 14 sets of $\overline{K}_1\mathbf{K}_4$. This completes the proof. $\square$

Table 4. Classes of $K_1 \cap K_4$ and not belonging to K_{10}

N		B	K_2	K_3	K_5	K_6	K_7	K_8	K_9	K_{13}
35	V_{15}	(000-), (-00-)				1		1		1
36	V_{23}	(000-)	1			1		1		1
37	T_{01}	(0001)	1		1		1			
38	V_{14}	(011-)	1			1				1
39	U_8	(0001), (0 - 01)	1		1					
40	V_7^*	(0001), (-001)			1					
41	V_7	(0001), (000-)	1							
42	V_6	(000-), (-10-)				1				1

N		B	K_2	K_3	K_5	K_6	K_7	K_8	K_9	K_{13}
43	V_{23}^*	(-111)		1	1				1	1
44	V_{22}	(-00-)		1		1		1		1
45	V_{22}^*	(-11-)		1		1			1	1
46	V_{15}^*	(-111), (-11-)		1					1	1
47	V_{14}^*	(-001)		1	1					1
48	U_7	(-00-), (-10-)		1		1				1
49	V_6^*	(-001), (-00-)		1						1
50	U_1	(0001), (-00-)								
51	U_3	(-001), (000-)								1

Theorem 3.3. *The number of E-closed classes of H_2 is 78.*

Proof. It follows from Theorem 3.1 and Theorem 3.2. $\square$

Table 5. Classes of $K_1 \cap K_4$ and belonging to K_{10}

N		B	K_2	K_3	K_5	K_6	K_7	K_8	K_9	K_{11}	K_{12}	K_{13}
52	U_{16}	(---)		1		1		1	1	1	1	1
53	S_{01}	(0101)	1		1		1			1		
54	V_{25}^*	(--11)		1	1				1		1	1
55	U_{15}	(0--1)	1		1					1		
56	U_{14}	(-10-)		1		1				1		1
57	V_{25}	(0-0-)	1			1		1			1	1
58	V_{24}	(--0-)		1		1		1			1	1
59	V_{24}^*	(--1-)		1		1			1		1	1
60	V_{21}^*	(--01)		1	1						1	1
61	V_{21}	(0-1-)	1			1					1	1
62	U_{13}	(--0-), (--1-)		1		1					1	1
63	U_{12}	(0101), (-10-)								1		
64	V_{19}^*	(--11), (--1-)		1					1		1	1
65	V_{19}	(0-0-), (--0-)				1		1			1	1
66	V_{18}^*	(-101)		1	1							1
67	U_{11}	(0-01)	1		1							
68	V_{18}	(010-)	1			1						1
69	U_{10}	(-10-), (--0-)		1		1						1
70	V_{12}^*	(--01), (--0-)		1							1	1
71	V_{12}	(0-0-), (--1-)				1					1	1
72	V_{11}^*	(0101), (-101)			1							
73	V_{11}	(0101), (010-)	1									
74	V_{10}^*	(-101), (-10-)		1								1
75	V_{10}	(010-), (-10-)				1						1
76	U_6	(--01), (0-0-)									1	1
77	U_5	(-101), (010-)										1
78	U_2	(0101), (--0-)										

References

- [1] D.Lau, Function algebras on finite sets. A basic course on many-valued logic and clone theory, 2006.
- [2] H.Machida, *Multiple-Valued Logic*, **8**(2002), 495–501.
- [3] H.Machida, J.Pantovic, On Maximal Hyperclones on $\{0, 1\}$ — a new approach, Proceedings of 38th IEEE International Symposium on Multiple-Valued Logic, 2008, 32–37.
- [4] B.A.Romov, *Multiple-Valued Logic*, **3**(1998), 285–300.
- [5] V.I.Panteleyev, *Vestnik Sam. Gos. Univ.*, **68**(2009), 60–79 (in Russian).
- [6] S.V.Zamaratskaya, V.I.Panteleyev, *The Bulletin of Irkutsk State University. Series Mathematics*, **15**(2016), 26–37 (in Russian).
- [7] S.V.Zamaratskaya, V.I.Panteleyev, *The Bulletin of Irkutsk State University. Series Mathematics*, **16**(2016), 58–70 (in Russian).

- [8] S.S.Marchenkov, *Discrete Math. Appl.*, **9**(1999), no. 6, 563–581.
DOI: 10.1515/dma.1999.9.6.563
- [9] S.S.Marchenkov, *Vestnik Moskov. Univ. Ser. 1. Mat. Mekh.*, 2003, no. 6, 37–39 (in Russian).
- [10] S.S.Marchenkov, *Discrete Math. Appl.*, **18**(2008), no. 4, 381–389.
- [11] S.S.Marchenkov, *Mathematical problems in cybernetics*, (2013), no. 19, 227–238 (in Russian).
- [12] S.A.Matveev, *Mathematical problems in cybernetics*, (2013), no.18, 239–244 (in Russian).
- [13] L.V.Ryabets, *The Bulletin of Irkutsk State University. Series Mathematics*, **17**(2016), 46–61 (in Russian).
- [14] L.V.Ryabets, *Intelligent systems. Theory and applications*, **20**(2016), no. 3, 79–84 (in Russian).
- [15] V.I.Panteleyev, L.V.Ryabets, *The Bulletin of Irkutsk State University. Series Mathematics*, **10**(2014), 93–105 (in Russian).

E-замкнутые классы гиперфункций ранга 2

Владимир И. Пантелеев

Леонид В. Рябец

Иркутский государственный университет

Иркутск, Российская Федерация

Аннотация. Гиперфункции представляют собой функции, задаваемые на конечном множестве и возвращающие в качестве своих значений все непустые подмножества рассматриваемого множества. В работе изложена классификация гиперфункций, заданных на двухэлементном множестве, относительно оператора *E*-замыкания. *E*-замкнутыми множествами гиперфункций являются множества, замкнутые относительно суперпозиции, оператора замыкания с разветвлением по предикату равенства, отождествления переменных и добавления фиктивных переменных. Показано, что рассматриваемая классификация приводит к конечному множеству замкнутых классов. В работе описаны все 78 *E*-замкнутых классов гиперфункций, среди которых есть 28 пар двойственных классов и 22 самодвойственных класса. Построена диаграмма включений классов, и для каждого класса указана его порождающая система.

Ключевые слова: замыкание, предикат равенства, гиперфункция, замкнутое множество, суперпозиция.