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Difference Equations and Generating Functions for some Lattice Path Problems

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An identity for generating functions is proved in this paper. A novel method to compute the number of restricted lattice paths is developed on the basis of this identity. The method employs a difference equation with non-constant coefficients. Dyck paths, Schröder paths, Motzkins path and other paths are computed to illustrate this method.

Keywords: difference equation, generating function, lattice path.

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For $z = (z_1, \dots, z_N)$ we introduce the ring of polynomials $\mathbb{C}[z]$, the field of rational functions $\mathbb{C}(z)$ and $\mathbb{C}[[z]]$ - the ring of formal power series in $z_1, \dots, z_N$, where $z^x = z_1^{x_1} \dots z_N^{x_N}$. If f is a function on $\mathbb{Z}_{\geq}^N$ it is identified as a function on $\mathbb{Z}^N$ by setting it equal to zero on the complement $\mathbb{Z}^N \setminus \mathbb{Z}_{\geq}^N$.

A linear finite difference equation is an equation of the form

$$\sum_{y \in S} c_y(x) f(x - y) = g(x), \quad x \in \mathbb{Z}^N, \quad (1)$$

where $S \subset \mathbb{Z}^N$ is finite, $c_y : \mathbb{Z}^N \rightarrow \mathbb{C}$ are a set of coefficient functions and $g : \mathbb{Z}^N \rightarrow \mathbb{C}$. A solution of (1) is a function $f : \mathbb{Z}^N \rightarrow \mathbb{C}$ that satisfies the equation.

Let $\ell \geq 0$. A lattice path with length ℓ is a finite sequence $p(0), p(1), \dots, p(L)$ of points in $\mathbb{Z}^N$, and its steps are the set of lattice vectors $\{0\} \cup \{p(k) - p(k-1) : k = 1, \dots, \ell\}$. Specific classes of lattice paths arise by imposing some conditions on the paths: the steps are in a specified $S \subset \mathbb{Z}^N$, the points are in a specified $P \subset \mathbb{Z}^N$, the length L is fixed, and the points are distinct (non intersecting paths).

In the context of lattice path counting problems the function $f : \mathbb{Z}^N \rightarrow \mathbb{Z}_{\geq}$ that counts the number $f(x)$ of paths in a specified class for which $p(0) = 0$ and $p(L) = x$ is computed (the condition $p(0) = 0$ does not result in a loss of generality).

In the context of restricted lattice class problems function f is computed for a class of paths whose points belong to a specified subset $P \subset \mathbb{Z}^N$. Clearly $0 \in P$ otherwise there are no paths in P that start at 0. If the possible set of steps $S \subset \mathbb{Z}^N$ then counting function f has support in P , $f(0) = 1$, and f satisfies the linear homogeneous difference equation

$$f(x) = \sum_{y \in S} \chi_P(x) f(x - y), \quad (2)$$

where χ_P is the characteristic function of P . We recall that $\chi_P(x) = 1$ if $x \in P$ and $\chi_P(x) = 0$ if $x \notin P$.

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In this paper function f is computed for selected classes of lattice paths. For these classes $S \subset \mathbb{Z}_{\geq}^N$ so f is supported on $\mathbb{R}_{\geq}^N$, and therefore f is uniquely represented by its *generating function* $F(z) \in \mathbb{C}[[z]]$. It is defined by

$$F(z) = \sum_{x \in \mathbb{Z}_{\geq}^N} f(x) z^x. \quad (3)$$

The proposed method employs a difference equation with non-constant coefficients for $f(x)$ to compute its generating function $F(z)$. This method is illustrated by counting Dyck paths, Schröder paths, Motzkin paths and more general paths. For these cases the terms in $F(z)$ that correspond to the terms with non-constant coefficients in the difference equation for $f(x)$ are the generating functions of a diagonal subsequence of $f(x)$.

Let $P(z) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha$ be a polynomial in $z \in \mathbb{C}^N$ and $x, m, \alpha \in \mathbb{Z}_{\geq}^N$. The inequality $0 \leq \alpha \leq m$ means that $0 \leq \alpha_j \leq m_j$ for all $j = 1, \dots, N$. Let us introduce $F_\alpha(z) = \sum_{x \geq \alpha} f(x) z^x$ and $\Phi_\alpha(z) = F(z) - F_\alpha(z)$, where the inequality $x \not\geq \alpha$ means that for at least one $j_0 \in \{1, \dots, N\}$ the inequality $x_{j_0} < \alpha_{j_0}$ holds.

Let δ_j be a shift operator over j^{th} variable: $\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_N)$. Then $\delta^\alpha = \delta_1^{\alpha_1} \circ \dots \circ \delta_n^{\alpha_n}$ and $P(\delta) = \sum_{0 \leq \alpha \leq m} c_\alpha \delta^\alpha$ is a polynomial difference operator with constant coefficients.

First a general identity for the generating functions is derived. Let us note that this theorem generalize the identity for generating functions given in [1].

Theorem 1. For any $F(z) \in \mathbb{C}[[z]]$ the identity

$$P(z)F(z) - \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) = \sum_{x \geq m} P(\delta^{-I})f(x)z^x \quad (4)$$

holds, where $I = (1, \dots, 1)$.

Proof. Let $F(z) = \sum_{x \geq 0} f(x) z^x$ and $P(z) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha$. Let us consider the product

$$\begin{aligned} P(z) \cdot F(z) &= \left(\sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \right) \left(\sum_{x \geq 0} f(x) z^x \right) = \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \left(\sum_{x \geq m-\alpha} f(x) z^x + \sum_{x \not\geq m-\alpha} f(x) z^x \right) = \\ &= \sum_{0 \leq \alpha \leq m} c_\alpha \sum_{x \geq m-\alpha} f(x) z^{x+\alpha} + \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x = \\ &= \sum_{0 \leq \alpha \leq m} c_\alpha \sum_{x \geq m} f(x-\alpha) z^x + \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x = \\ &= \sum_{x \geq m} \left(\sum_{0 \leq \alpha \leq m} c_\alpha f(x-\alpha) \right) z^x + \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \sum_{x \not\geq m-\alpha} f(x) z^x. \end{aligned}$$

Since $\sum_{0 \leq \alpha \leq m} c_\alpha f(x-\alpha) = P(\delta^{-I})f(x)$ and $\Phi_{m-\alpha}(z) = \sum_{x \not\geq m-\alpha} f(x) z^x$ we obtain (4). $\square$

Identity (4) implies that for any initial data $\varphi(x)$, $x \not\geq m$, $x \geq 0$ and any function $g(x)$, $x \geq m$ the equation $P(\delta^{-I})f(x) = g(x)$ has a unique solution $f(x)$ that satisfies initial data: $f(x) = \varphi(x)$, $x \geq 0$, $x \not\geq m$ (see [2, 3, 4]). If $G(z) = \sum_{x \geq m} g(x) z^x$ then identity (4) gives

$$F(z) = \frac{1}{P(z)} \sum_{0 \leq \alpha \leq m} c_\alpha z^\alpha \Phi_{m-\alpha}(z) + \frac{G(z)}{P(z)}.$$

Let $\Delta = \{e^1, \dots, e^N\}$, where vector $e^j = (0, \dots, 0, 1, 0, \dots, 0)$ contains one on the j^{th} place for $j = 1, \dots, N$. Let $f(x)$ be the number of paths from the origin to the point $x \in \mathbb{Z}_{\geq 0}^N$.

Corollary. *If $f(x)$ is the number of lattice paths from the origin to $x \in \mathbb{Z}_{\geq 0}^N$ using steps from the set Δ then its generating function $F(z)$ is*

$$F(z) = \frac{1}{1 - z_1 - \dots - z_N}.$$

Proof. Let us note that function $f(x)$ satisfies the basic recurrence relation $f(x) = f(x - e^1) + \dots + f(x - e^N)$ which implies that the right side of identity (4) is equal to 0.

Let us write identity (4) for the two dimensional case:

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_2)F(0, z_2) - (1 - z_1)F(z_1, 0) + F(0, 0) = 0.$$

Since $f(x_1, 0) = f(0, x_2) = 1$ for all non-negative integers x_1 and x_2 we obtain

$$F(0, z_2) = \frac{1}{1 - z_2}, \quad F(z_1, 0) = \frac{1}{1 - z_1}, \quad F(0, 0) = 1.$$

$$\text{Then } F(z_1, z_2) = \frac{1}{1 - z_1 - z_2}.$$

For $N = 3$ we have

$$\begin{aligned} & (1 - z_1 - z_2 - z_3)F(z_1, z_2, z_3) - (1 - z_1 - z_2)F(z_1, z_2, 0) - (1 - z_1 - z_3)F(z_1, 0, z_3) - \\ & - (1 - z_2 - z_3)F(0, z_2, z_3) + (1 - z_1)F(z_1, 0, 0) + (1 - z_2)F(0, z_2, 0) + \\ & + (1 - z_3)F(0, 0, z_3) - F(0, 0, 0) = 0. \end{aligned}$$

Considering the three dimensional case, we obtain

$$F(z_1, z_2, z_3) = \frac{1}{1 - z_1 - z_2 - z_3}.$$

Repeating this process, one can obtain the generating function for any $N > 1$. $\square$

Let us demonstrate another way of using identity (4) for the two-dimensional case which is useful for some lattice path problems.

Let us introduce $F(z_1, z_2) = \sum_{(x_1, x_2) \geq (0, 0)} f(x_1, x_2) z_1^{x_1} z_2^{x_2}$ and $F_{p,q}(z_1, z_2) = \sum_{k=1}^{\infty} f(pk, qk) z_1^{pk} z_2^{qk}$, where $(p, q) \in \mathbb{Z}_{\geq 0}^2$.

Let us assume that function $f(x, y) = \varphi(x, y)$, $(x, y) \not\geq (p, q)$, $(x, y) \geq 0$ satisfies the difference equation

$$P(\delta_1^{-1}, \delta_2^{-1})f(x, y) = g(x, y),$$

where

$$g(x, y) = \begin{cases} f(x, y), & \text{if } x = pk, y = qk, k \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $F_{p,q}(z_1, z_2) = \sum_{(x, y) \geq (p, q)} g(x, y) z_1^x z_2^y$ and identity (4) becomes

$$P(z_1, z_2)F(z_1, z_2) - \sum_{\substack{0 \leq \alpha_1 \leq p \\ 0 \leq \alpha_2 \leq q}} c_{\alpha_1, \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2} \Phi_{p-\alpha_1, q-\alpha_2}(z_1, z_2) = F_{p,q}(z_1, z_2).$$

If $z_1 = z_1(t)$, $z_2 = z_2(t)$ is a solution of system $\begin{cases} z_1^p z_2^q = t \\ P(z_1, z_2) = 0 \end{cases}$ then function $F_{p,q}$ is

$$F_{p,q}(z_1(t), z_2(t)) = - \sum_{\substack{0 \leq \alpha_1 \leq p \\ 0 \leq \alpha_2 \leq q}} c_{\alpha_1, \alpha_2} z_1^{\alpha_1}(t) z_2^{\alpha_2}(t) \Phi_{p-\alpha_1, q-\alpha_2}(z_1(t), z_2(t)).$$

Let us consider examples of some well-known lattice paths: Dyck, Motzkin and Schröder paths (see [2, 5, 6, 7, 8, 9, 10]). A linear transformation is used to map the mentioned above lattice paths to the lattice path in $\mathbb{Z}_{\geq}^2$. It allows one to use methods for finding generating functions [3] and [11]. However, to study lattice paths on or over a rational slope linear difference equations with non-constant coefficients are used to put restrictions on them.

Dyck paths start at the origin and stay on or above the main diagonal $y = x$ (see [2, 6, 12]). They use steps $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to (x, y) . The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x-1, y) - f(x, y-1) = -\delta_0(x-y-1)f(x-1, y), \quad (5)$$

where $\delta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$ is the Kronecker symbol. The initial data are

$$f(x, 0) = 0, \quad x = 1, 2, \dots, \quad f(0, y) = 1, \quad y = 0, 1, 2, \dots \quad (6)$$

Let $F_{11}(t)$ be a diagonal power series of $F(z_1, z_2)$

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k, k)t^k.$$

Proposition 1. *Let $F(z_1, z_2)$ be the generating function of the solution of equation (5). Then the series $F(z_1, z_2)$ satisfies the following functional equation*

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k)(z_1 z_2)^k.$$

If $f(x, y)$ satisfies initial conditions (6) then we obtain a diagonal power series

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k, k)t^k = \frac{1 - 2t - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \dots \quad (7)$$

Proof. For $N = 2$ and $P(z_1, z_2) = 1 - z_1 - z_2$ we have $c_{00} = 1$, $c_{10} = c_{01} = -1$, $c_{11} = 0$, $m = (1, 1)$, $\Phi_{1,1}(z_1, z_2) = F(z_1, 0) + F(0, z_2) - F(0, 0)$, $\Phi_{1,0}(z_1, z_2) = F(0, z_2)$, $\Phi_{0,1}(z_1, z_2) = F(z_1, 0)$, $\Phi_{0,0}(z_1, z_2) = 0$.

Then by theorem we obtain

$$\begin{aligned} (1 - z_1 - z_2)F(z_1, z_2) - c_{00}\Phi_{1,1}(z_1, z_2) - c_{10}z_1\Phi_{0,1}(z_1, z_2) - c_{01}z_2\Phi_{1,0}(z_1, z_2) = \\ = \sum_{\substack{x \geq 1 \\ y \geq 1}} (1 - \delta_1^{-I} - \delta_2^{-I})f(x, y)z_1^x z_2^y. \end{aligned}$$

Using difference equation (5), we have

$$(1 - z_1 - z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k)(z_1 z_2)^k,$$

where $F(0, z_2) = \sum_{y \geq 0} f(0, y) z_2^y = \frac{1}{1 - z_2}$, $F(z_1, 0) = \sum_{x \geq 0} f(x, 0) z_1^x = 1$, $F(0, 0) = f(0, 0) = 1$.

Let $P(z_1, z_2) = 1 - z_1 - z_2 = 0$. Then we obtain

$$-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k) (z_1(1 - z_1))^k.$$

Let us introduce $t = z_1(1 - z_1)$. Then $z_1 = \frac{1 + \sqrt{1 - 4t}}{2}$. After expansion of $\frac{1}{z_1}$ we obtain

$$\sum_{k=1}^{\infty} f(k, k) t^k = -1 + \frac{1 - \sqrt{1 - 4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 \dots$$

It proves the proposition. $\square$

The coefficients of series (7) represent the Catalan numbers $f(k, k) = \frac{1}{k-1} \binom{2k}{k}$, $k \geq 1$.

Schröder paths start at the origin and stay on or above the main diagonal $y = x$ (see [2]) using steps $(1, 0)$, $(0, 1)$, $(1, 1)$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to (x, y) . The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x - 1, y) - f(x, y - 1) - f(x - 1, y - 1) = -\delta_0(x - y - 1) f(x - 1, y) \quad (8)$$

with the initial data

$$f(x, 0) = 0, \quad x = 1, 2, \dots, \quad f(0, y) = 1, \quad y = 0, 1, 2, \dots \quad (9)$$

Proposition 2. Let $F(z_1, z_2)$ be the generating function of the solution of equation (8). Then the series $F(z_1, z_2)$ satisfies the following functional equation

$$(1 - z_1 - z_2 - z_1 z_2) F(z_1, z_2) - (1 - z_1) F(z_1, 0) - (1 - z_2) F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k) (z_1 z_2)^k.$$

If $f(x, y)$ satisfies initial conditions (9) then we obtain a diagonal power series

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k, k) t^k = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{2t} = 2t + 6t^2 + 22t^3 + 90t^4 + \dots \quad (10)$$

Proof. Using theorem and difference equation (8), we have

$$(1 - z_1 - z_2 - z_1 z_2) F(z_1, z_2) - (1 - z_1) F(z_1, 0) - (1 - z_2) F(0, z_2) + F(0, 0) = -z_1 \sum_{k \geq 1} f(k, k) (z_1 z_2)^k,$$

where $F(0, z_2) = \sum_{y \geq 0} f(0, y) z_2^y = \frac{1}{1 - z_2}$, $F(z_1, 0) = \sum_{x \geq 0} f(x, 0) z_1^x = 1$, $F(0, 0) = f(0, 0) = 1$.

Let $P(z_1, z_2) = 1 - z_1 - z_2 - z_1 z_2 = 0$, we obtain

$$-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k) \left(\frac{z_1(1 - z_1)}{1 + z_1} \right)^k.$$

Let us introduce $t = \frac{z_1(1 - z_1)}{1 + z_1}$. Then $z_1 = \frac{1 - t + \sqrt{1 - 6t + t^2}}{2}$. After expansion of $\frac{1}{z_1}$ we obtain (10). $\square$

Coefficients of series (10) coincide with the numbers of the Schröder paths ending on the main diagonal $y = x$.

Motzkin paths start at the origin and stay on or above the main diagonal $y = x$ (see [8]) using steps $(2, 0), (0, 2), (1, 1)$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to (x, y) . The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x-2, y) - f(x, y-2) - f(x-1, y-1) = -(\delta_0(x-y-1) + \delta_0(x-y-2))f(x-2, y) \quad (11)$$

with the initial data

$$\begin{aligned} f(x, 0) = 0, \quad x = 1, 2, 3, \dots, \quad f(0, y) = \frac{1 + (-1)^y}{2}, \quad y = 0, 1, 2, \dots, \\ f(x, 1) = 0, \quad x = 2, 3, 4, \dots, \quad f(1, y) = \frac{(1 - (-1)^y)(y+1)}{4}, \quad y = 1, 2, 3, \dots \end{aligned} \quad (12)$$

Proposition 3. Let $F(z_1, z_2)$ be the generating function of the solution of equation (11). Then the series $F(z_1, z_2)$ satisfies the following functional equation

$$(1 - z_1^2 - z_2^2 - z_1 z_2)F(z_1, z_2) - (1 - z_1)F(z_1, 0) - (1 - z_2)F(0, z_2) + F(0, 0) = -z_1^2 \sum_{k \geq 2} f(k, k)(z_1 z_2)^k.$$

If $f(x, y)$ satisfies initial conditions (12) then we obtain a diagonal power series

$$F_{11}(t) = \sum_{k=2}^{\infty} f(k, k)t^k = -1 - t + \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2} = 2t^2 + 4t^3 + 9t^4 + 21t^5 + \dots \quad (13)$$

Proof. Using theorem and difference equation (11), we have

$$\begin{aligned} (1 - z_1^2 - z_2^2 - z_1 z_2)F(z_1, z_2) - (1 - z_1^2 - z_1 z_2)F(z_1, 0) - (1 - z_2^2 - z_1 z_2)F(0, z_2) + \\ + F(0, 0)(1 - z_1 z_2) - \tilde{\Phi}_{1,0}(z_1, z_2)(1 - z_2^2) - \tilde{\Phi}_{0,1}(z_1, z_2)(1 - z_1^2) + f(1, 1)z_1 z_2 = \\ = -z_1^2 \sum_{k \geq 2} f(k, k)(z_1 z_2)^k, \end{aligned}$$

where $F(z_1, 0) = 1$, $F(0, z_2) = \frac{1}{1 - z_2^2}$, $F(0, 0) = 1$, $\tilde{\Phi}_{0,1}(z_1, z_2) = z_1 z_2$, $\tilde{\Phi}_{1,0}(z_1, z_2) = \frac{z_1 z_2}{(1 - z_2^2)^2}$, $f(1, 1) = 1$.

Let $P(z_1, z_2) = 1 - z_1^2 - z_2^2 - z_1 z_2 = 0$. Then we obtain

$$-z_1 z_2 - 1 + \frac{1}{z_1^2} = \sum_{k=2}^{\infty} f(k, k) \left(\frac{z_1(\sqrt{4 - 3z_1^2} - z_1)}{2} \right)^k.$$

Let us introduce $t = z_1 z_2$. Then $z_1^2 = \frac{1 - t + \sqrt{1 - 2t - 3t^2}}{2}$. After expansion of $\frac{1}{z_1^2}$ we obtain (13). $\square$

Coefficients of series (13) coincides with the numbers of the Motzkin paths ending on the main diagonal $y = x$.

Let us consider some generalization of the problem of enumerating lattice paths with steps $(1, 0), (0, 1), (r, r)$ which start at the origin and stay on or above the main diagonal $y = x$. Let $f(x, y)$ be the number of paths going from $(0, 0)$ to (x, y) . The number of paths $f(x, y)$ satisfies the difference equation

$$f(x, y) - f(x-1, y) - f(x, y-1) - f(x-r, y-r) = -\delta_0(x-y-1)f(x-1, y) \quad (14)$$

with some initial data

$$f(x, y) = \varphi(x, y), \quad (x, y) \geq (0, 0), \quad (x, y) \not\geq (r, r). \quad (15)$$

Proposition 4. *Let $F(z_1, z_2)$ be the generating function of the solution of equation (14). Then the series $F(z_1, z_2)$ satisfies the following functional equation*

$$(1 - z_1 - z_2 - z_1^r z_2^r)F(z_1, z_2) - \Phi_{r,r}(z_1, z_2) + z_1 \Phi_{r-1,r}(z_1, z_2) + z_2 \Phi_{r,r-1}(z_1, z_2) = -z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k.$$

Proof. Using theorem and difference equation (14), we have

$$\begin{aligned} (1 - z_1 - z_2 - z_1^r z_2^r)F(z_1, z_2) - \Phi_{r,r}(z_1, z_2) + z_1 \Phi_{r-1,r}(z_1, z_2) + z_2 \Phi_{r,r-1}(z_1, z_2) = \\ = -z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k, \end{aligned}$$

where

$$\begin{aligned} \Phi_{r,r}(z_1, z_2) &= \Phi_{r-1,r-1}(z_1, z_2) + \tilde{\Phi}_{0,r-1}(z_1, z_2) + \tilde{\Phi}_{r-1,0}(z_1, z_2) - f(r-1, r-1)(z_1 z_2)^{r-1}, \\ \Phi_{r-1,r}(z_1, z_2) &= \Phi_{r-1,r-1}(z_1, z_2) + \tilde{\Phi}_{0,r-1}(z_1, z_2), \\ \Phi_{r,r-1}(z_1, z_2) &= \Phi_{r-1,r-1}(z_1, z_2) + \tilde{\Phi}_{r-1,0}(z_1, z_2), \\ \Phi_{r-1,r-1}(z_1, z_2) &= \sum_{i=0}^{r-2} \tilde{\Phi}_{i,0}(z_1, z_2), \\ \tilde{\Phi}_{0,r-1}(z_1, z_2) &= \sum_{x=r-1}^{\infty} f(x, r-1) z_1^x z_2^{r-1}, \quad \tilde{\Phi}_{r-1,0}(z_1, z_2) = \sum_{y=r-1}^{\infty} f(r-1, y) z_1^{r-1} z_2^y. \end{aligned}$$

Let $P(z_1, z_2) = 1 - z_1 - z_2 - z_1^r z_2^r = 0$. Then we obtain

$$\begin{aligned} \Phi_{r-1,r-1}(z_1, z_2) + \tilde{\Phi}_{0,r-1}(z_1, z_2) + \tilde{\Phi}_{r-1,0}(z_1, z_2) - f(r-1, r-1)(z_1 z_2)^{r-1} - z_1(\Phi_{r-1,r-1}(z_1, z_2) + \\ + \tilde{\Phi}_{0,r-1}(z_1, z_2)) - z_2(\Phi_{r-1,r-1}(z_1, z_2) + \tilde{\Phi}_{r-1,0}(z_1, z_2)) = z_1 \sum_{k \geq r} f(k, k)(z_1 z_2)^k. \end{aligned}$$

Since $f(x, y) = 0$ below the diagonal, we obtain

$$(1 - z_1 - z_2)\Phi_{r-1,r-1}(z_1, z_2) + (1 - z_2)\tilde{\Phi}_{r-1,0}(z_1, z_2) = z_1 \sum_{k \geq r-1} f(k, k)(z_1 z_2)^k.$$

Let us introduce $t = z_1 z_2$ implies $z_1 = \frac{1 - t^r + \sqrt{1 - 4t - 2t^r + t^{2r}}}{2}$. Then

$$\frac{1}{z_1}((1 - z_1 - z_2)\Phi_{r-1,r-1}(z_1, z_2) + (1 - z_2)\tilde{\Phi}_{r-1,0}(z_1, z_2)) = \sum_{k \geq r-1} f(k, k)t^k. \quad (16)$$

For $r = 2$ we have the initial data $f(x, 0) = 0$, $x = 1, 2, 3, \dots$, $f(0, y) = 1$, $y = 0, 1, 2, \dots$, $f(x, 1) = 0$, $x = 2, 3, 4, \dots$, $f(1, y) = y$, $y = 1, 2, 3, \dots$

Using (16), we obtain

$$-1 + \frac{1}{z_1} = \sum_{k=1}^{\infty} f(k, k)(z_1 z_2)^k.$$

Let us introduce $t = z_1 z_2$. Then $z_1 = \frac{1 - t^2 + \sqrt{1 - 4t - 2t^2 + t^4}}{2}$. After expansion of $\frac{1}{z_1}$, we obtain

$$-1 + \frac{1}{z_1} = -1 + \frac{1 - t^2 - \sqrt{1 - 4t - 2t^2 + t^4}}{2t} = t + 3t^2 + 8t^3 + 25t^4 + 83t^5 + \dots \quad (17)$$

It proves the proposition. $\square$

Coefficients of series (17) coincides with the numbers of other paths ending on the main diagonal $y = x$.

References

- [1] A.P.Lyapin, S.Chandragiri, Generating functions for vector partitions and a basic recurrence relation, *Journal of Difference Equations and Applications*, **25**(2019), 1052–1061.
- [2] M.Bousquet-Mélou, M.Petkovšek, Linear recurrences with constant coefficients: the multivariate case, *Discrete Mathematics*, **225**(2000), 51–75.
- [3] E.K.Leinartas, A.P.Lyapin, On the rationality of multidimensional recursive series, *J. Sib. Fed. Univ. Math. Phys*, **4**(2009), no. 2, 449–455.
- [4] E.K.Leinartas, Multiple Laurent series and fundamental solutions of linear difference equations, *Siberian Mathematical Journal*, **48**(2007), no. 2, 268–272.
- [5] J.Labelle, Y.N.Yeh, Generalized Dyck paths, *Discrete Math*, **82**(1990), 1–6.
- [6] D.Merlini, R.Sprugnoli, M.C.Verri, The area determined by underdiagonal lattice paths, *Trees in Algebra and Programming — CAAP*, 1996, 59–71.
- [7] I.M.Gessel, A factorization for formal Laurent series and lattice path enumeration, *J. Combin. Theory. Ser. A*, **28**(1980), 321–337.
- [8] A.Dvoretzky, Th.Motzkin, A problem of arrangements, *Duke Math. J.*, **14**(1947), 305–313.
- [9] P.Duchon, On the enumeration and generation of generalized Dyck words, *Discrete Math*, **225**(2000), 121–135.
- [10] R.Stanley, *Enumerative Combinatorics*, Vol. 1, 1990.
- [11] A.A.Kytmanov, A.P.Lyapin, T.M.Sadykov, Evaluating the rational generating function for the solution of the Cauchy problem for a two-dimensional difference equation with constant coefficients, *Programming and computer software*, **43**(2017), no. 2, 105–111.
- [12] D.Merlini, D.G.Rogers, R.Sprugnoli, M.C.Verri, Underdiagonal lattice paths with unrestricted steps, *Discrete Applied Mathematics*, **91**(1999), 197–213.

Разностные уравнения и производящие функции в некоторых задачах о решеточных путях

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В работе доказано тождество для производящих функций, на основе которого разработан метод вычисления числа путей на целочисленной решетке с ограничениями. Данный метод использует разностные уравнения с переменными коэффициентами. В качестве примеров вычислены производящие функции для путей Дика, Моцкина и Шрёдера.

Ключевые слова: разностные уравнения, производящие функции, решеточные пути.