Non-Transitive Temporal Multi-Agent Logic, Information and Knowledge, Deciding Algorithms

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Abstract

Multi-agent and temporal logics are active domains in Information Sciences, CS and AI. Attention has predominantly focused on logics based at transitive relational models, with particular emphasis on transitive time. This however does not seem to be a very reliable assumption. Non-transitivity of passing information may be demonstrated with relative ease through persuasive examples. Therefore, in this paper, we introduce and study multi-agent temporal logics based at linear non-transitive time. The one more innovative step is consideration of incomplete information: the information/knowledge with lacunas, – the linear time with forgettable intervals of time in past. Technically, the most important problems are problems of satisfiability and decidability of suggested logics. The main results are found by us algorithms which compute satisfiability and solve decidability (and so provide solutions to these problems). The paper concludes by posing a series of open problems.

Keywords: temporal logic, computability, information, multi-agent logic, satisfiability, decidability, deciding algorithms, non-transitive time

1 Introduction

Information and knowledge are inherent within Computer Science (CS) and Information Sciences (IS). Computational aspects analyzing knowledge (often referred to as computational intelligence) have formed a solid branch within CS and IS (using as a base instruments from mathematics and models of symbolic computation). The more technical aspects are often occupied with the construction of efficient algorithms for handling information, retrieving new information from known facts (these constructions are reasonably often implemented via different proof systems, tableau, robust information systems).

In *Reasoning about Knowledge, MIT press, 1995 [5],* Fagin, et al, presumably first time, in a very systematic manner, summaries the modeling of knowledge interpreted via the multi-agent's approach based at relational Kripke-Hintikka like models. An essential feature of this approach is the usage of the multi-agent's logic with agent's knowledge operations K_i behaving as $S5$- modalities. Another efficient tool from this book was temporal logic, which is easy to comprehend since knowledge is collected in a time environment, an important resource. In turn, temporal logic is a popular area in CS and AI (cf. e.g. Gabbay and Hodkinson [6, 7, 8]). Linear temporal logic $\mathcal{LTL}$ (with Until and Next) was introduced by Manna and Pnueli [12, 13]; efficient tools for satisfiability in $\mathcal{LTL}$ based at automaton theory were suggested by Vardi [24, 25]; $\mathcal{LTL}$ found usage for analyzing protocols of computations and verification of consistency, compatibility and other characteristics from CS.

Starting from the objective general multi-disciplinary environment, knowledge was often analyzed via multi-agent techniques, various agent's qualities, e.g. interaction or autonomy, effects of cooperation, etc were investigated (cf. e.g. Wooldridge et al [27, 28, 29], Lomuscio et al [11, 3]). For mathematical description of information and knowledge, tools of modal and other non-classical logics were efficiently used. This approach was created by Hintikka [10] (1962) in the book: *Knowledge and Belief*. Since then, the framework based at non classical logic has repeatedly demonstrated its efficiency (cf. e.g. Artemov [1], Balbiani, Vakarelov [4], Halpern [9], Vakarelov [26]). In particular, we earlier studied multi-agent's logic with distances, the satisfiability problem for it was solved (Rybakov et al [20]); models for the conception of Chance Discovery in multi-agents environment were developed (Rybakov [21, 22]); a logic modeling uncertainty via agent's views was investigated (cf. McLean et al [14]); representation of agents interaction (as a dual of the common knowledge - an elegant conception suggested and developed in Fagin et al [5]) was suggested in Rybakov [15, 18, 19].

The aim of this paper is construction of a logical framework for handling incomplete information and knowledge of agents (with lacunas, for instance, – with fragments of forgotten information in the past) in non-transitive temporal logic. The bases of our approach is suggested through mathematical (symbolic) models for the agent's knowledge; they use linear time distributed along transitive time-intervals (maximal amount of events which agents may remember) and the agent's time-accessibility relations. The latter may be not complete; agents may not remember the same amount of events, and so accessibility relations may be non-transitive. These models are generalizations of Kripke-Hintikka relational models (which are an efficient tool for modal and temporal logic, for non-classical logics overall).

To reason about properties of information, to evaluate statements of truth, we use a language of multi-agent's logic combined with the language of temporal logic. Chosen rules for computation truth values for statements (coded by formulas) handle the presence of lacunas in knowledge collected from past (so to say, – take to account forgotten past). We define the logic based on these models as the set of all formulas (in the chosen logical language) which are true for all models. Main problems that we study are satisfiability problem and decidability problem for this logic. We solve them via finding algorithms verifying satisfiability and solving decidability. In the final part of the paper, a version of our original logic with admission non-bounded time relation is considered; the problem of decidability is also solved.

The paper is organized as follows. In Section 2, we define the models (semantics) and introduce the language, define our logic MA^{Int}_{Lin} - multi-agent, linear, intransitive logic. The novelty here, in particular, is the fact that the time in these models is not transitive. We comment and justify chosen models, providing illustrative examples explaining why admission about transitivity of time is not always correct and how the presence of incomplete information may be handled in our approach. Section 3 develops mathematical techniques for our relational multi–agent models. We use approach of transformation the formulas to rules and a construction their reduced normal forms (which drastically simplifies constructions and proofs because these rules do not allow nested logical operations). The main content is Theorem 14 stating that the satisfiability problem for MA_{Lin}^{Int} is decidable (and hence the logic MA_{Lin}^{Int} is decidable). We find an algorithm which computes satisfiability of formulas, solves the decidability problem for MA_{Lin}^{Int}. Section 4 considers another version of the logic MA_{Lin}^{Int} - the one which allows insight to all the past (so to say with the one omniscient agent which remember all the past). We extend the techniques from Sections 3 and 4 and show that this logic also is decidable (Theorem 24). The paper concludes by posing a series of open problems.

2 Multi-Agent's Linear Temporal Logic Based at Non-transitive Time

Our aim is to built a logical framework for computation (evaluation) truth values of statements describing information and knowledge in multi-agent environment. We will offer models formalizing information being based at temporal states of computations, therefore the logical language we will apply forms an extension of the one for the temporal logic. The formulas of this language are built up from a (an infinite) set of propositional letters Pr, boolean logical operations $\wedge, \vee, \rightarrow, \neg$ and the set of temporal operations: unary operation N (next) and a set of binary temporal operations U_j (until for the agent j), where $j \in Ag$ (Ag is supposed to be a set of all agents under consideration). So, the language is an extension of the language of the standard temporal linear logic with a new finite set of logical operations of the sort until. The formation rules for formulas are as usual. That is: for any $p \in Pr$, p is a formula, if φ and ψ are formulas then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\neg\varphi$, $\varphi U_j \psi$ and $N\varphi$ are formulas.

We will illustrate a bit later flexibility of this language and its usage for computational intelligence. Now we introduce symbolic mathematical models — semantics for our logic.

Definition 1 *An* multi-agent, linear, non-transitive frame *is a tuple*

$$\mathcal{ALF}_{In} = \langle W_{\mathcal{ALF}_{In}}, (\bigcup_{\xi \in In, j \in [1,k]} \langle R_{\xi,j} \rangle), Nxt, \rangle, \text{ such that}$$

- $W_{\mathcal{ALF}_{In}} := \bigcup_{\xi \in In \subset N} It[\xi]$, *where* N *is the set of all natural numbers; for any* $\xi \in In$, $d(\xi) \in In, d(\xi) > \xi$ *and* $It[\xi]$ *is the closed interval of all natural numbers situated between* ξ *and* $d(\xi)$ *(i.e., in particular, containing also both* ξ *and* $d(\xi)$*), so* $It[\xi] = [\xi, d(\xi)]$*);*
- $\forall \xi_1, \xi_2 \in In$, $\xi_1 \neq \xi_2 \Rightarrow (\xi_1, d(\xi_1)) \cap (\xi_2, d(\xi_2)) = \emptyset$*;*

- *any* $R_{\xi,j}$ *is the restriction of the standard linear order* ($\leq$) *in the interval* $It[\xi]$ *on a subset* $Dom_{\xi,j} \subseteq It(\xi)$*;*
- $\forall j \in [1, k]$, $R_j := \bigcup_{\xi \in In} R_{\xi,j}$*; (note that then any* R_j *is non-transitive but linear);*
- Nxt *is the standard next relation on* N*:* $[n\ Nxt\ m]$ *iff* $m = n + 1$*; we will write* $Nxt(a) = b$ *to denote that* $b = a + 1$ *.*

Notice that we consider $(\bigcup_{\xi \in In, J \in [1,k]} \langle R_{\xi,j} \rangle)$ as not a binary relation, but as the infinite countable set of finite binary relations $R_{\xi,j}$. Any $d(\xi) \in In$ for any ξ is the measure of intransitivity, we interpret $[\xi, d(\xi)]$ as the maximal interval of time which agents may remember at time point ξ.

We will use notation $|\mathcal{ALF}_{In}|$ for the base set $W_{\mathcal{ALF}_{In}}$ of $\mathcal{ALF}_{In}$, also for short we will write $a \in \mathcal{ALF}_{In}$ to say that $a \in |\mathcal{ALF}_{In}|$.

My may, for example, consider the relations $R_{\xi,j}$ and $\leq$ actually directed to past, and $a \leq b$ would mean that b was earlier than a, b is past for a. We may consider the memory of agents which may have gaps, i.e. agents remember the past but not continuously. E.g. for an agent 1 at point 3 from $[3, d(3)]$, where $d(3) = 12$, $R_{3,1}$ may be $\leq$ on $[3, 4, 5] \cup [7, 9] \cup [11, 12]$, that is the agent does not remember time points 6, 8 and 10.

This approach may also interpret the case when the length of memory of all agents may be different (they may have different volume of storage for past). That works if we consider some gaps in relations $R_{\xi,j}$ on $[\xi, d(\xi)]$ immediately before $d(\xi)$. In particular it is admissible if all agents have gaps before $d(\xi)$. This means that potentially measure of intransitivity of time might be big, but all agents do not reach its limit – point $d(\xi)$.

As usual for temporal logic, for each $\mathcal{ALF}_{In}$ we may define a model by introduction a valuation V on $\mathcal{ALF}_{In}$ for a set of propositional letters p: $V(p) \subseteq W_{\mathcal{ALF}_{In}}$, and extend it to all formulas as follows:

Definition 2 For any $a \in W_{\mathcal{ALF}_{In}}$ and $j \in Ag$:

$$(\mathcal{ALF}_{In}, a) \Vdash_V p \iff p \in V(p);$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V \neg\varphi \iff (\mathcal{ALF}_{In}, a) \nVdash_V \varphi;$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V (\varphi \wedge \psi) \iff ((\mathcal{ALF}_{In}, a) \Vdash_V \varphi) \wedge ((\mathcal{ALF}_{In}, a) \Vdash_V \psi);$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V (\varphi \vee \psi) \iff ((\mathcal{ALF}_{In}, a) \Vdash_V \varphi) \vee ((\mathcal{ALF}_{In}, a) \Vdash_V \psi);$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V (\varphi \to \psi) \iff ((\mathcal{ALF}_{In}, a) \Vdash_V \psi) \vee ((\mathcal{ALF}_{In}, a) \nVdash_V \psi);$$

for formulas $\varphi U_j \psi$ we define the truth value as follows:

$$(\mathcal{ALF}_{In}, a) \Vdash_V (\varphi\ U_j\ \psi) \iff$$

$$\exists\xi(a \in It(\xi))\ \&\ \exists b[(aR_{\xi,j}b) \wedge ((\mathcal{ALF}_{In}, b) \Vdash_V \psi) \wedge$$

$$\forall c[(c \in Dom_{\xi,j}\ \&\ aR_{\xi,j}c\ \&\ c < b) \Rightarrow (\mathcal{ALF}_{In}, c) \Vdash_V \varphi]];$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V N\varphi \iff [(a\ Nxt\ b) \Rightarrow (\mathcal{ALF}_{In}, b) \Vdash_V \varphi].$$

$(\mathcal{ALF}_{In}, a) \Vdash_V \varphi$ to be read *the formula φ is true (valid)* at the state a w.r.t. the valuation V.

Definition 3 *The logic MA^{Int}_{Lin} is the set of all formulas which are valid in any model based at any $\mathcal{ALF}_{In}$ frame.*

We illustrate below why we admit that time might be non-transitive.

Standpoint (i). Individual human being view. If we interpret time in the past as a line of all events which we individually remember, and time flows as a chain of events which individuals know and pass to each other, the things are clear. We do not know and do not remember all what our ancestors knew.

Standpoint (ii). Computational view. Inspections of protocols for computations are limited by time resources and have non-uniform length (else, at any point of inspection, verification may refer to stored old protocols). Therefore, if we interpret our models as the ones reflecting verification of computations, the amount of available check points is finite, but not all of them might be in disposal for inspection in a given time point.

Standpoint (iii). Agent's-admins view. We may consider states (worlds of our model) as checkpoints of admins (agents) for inspections of behavior a network in past. Any admin has allowed amount of inspections for previous states, but only within the areas of its (his/her) responsibility (by security or another reasons). Thus, the accessibility is not transitive again. An admin (a1) can reach a state, and therein, an admin (a2) responsible for this state (it may be a new one or the same one as well) has again some allowed amount of inspections to the past. But, in total, (a1) cannot inspect all states accessible for (a2).

Standpoint (iv). Agent's-user's view. If we consider the states of models as the content of web pages available for users, any surf step is accessibility relation, and starting from any web page user may achieve, using links in hypertext(s), some foremost available web sites. The latter one may have web links which are only available for individuals possessing passwords for accessibility. Users having password may continue web surf longer. Clearly that in this approach, web surfing appears to be a non-transitive relation. That is, if we interpret web surf as time-steps, the accessibility is intransitive.

Standpoint (v). View on time in past for collecting knowledge. In human perception, only some finite intervals of time in past are available to individuals to inspect evens and to record knowledge collected until the current time state. The time in past, in our feelings, looks as linear and we have only a finite amount of memory to remember information and events. There, in past, at foremost available (memorable) time point, individuals again had a memorable interval of time with collected information, and so forth. So, time in the past, generally speaking, looks as not transitive if we interpret it via collecting knowledge.

Now we pause briefly to comment why we consider incomplete information: the knowledge of the agents with forgotten information in past.

(i) Let we interpret the agent's knowledge as the protocols of different concurrent computations: several parallel threads of a computation. Then, first, some protocols may contains an information received at a step i while others may miss it. That may happen by hardware failure or by specific behavior of the software for different computational threads. Thus, in interpretation of this situation, our $R_{\xi,j}$ may not cover all time interval $[\xi, d(\xi)]$, theirs domains $D_{\xi,j}$ may be smaller than $[\xi, d(\xi)]$ itself.

(ii) Similar comment may be refereed to human memory and records of different authors about historical events. Some others remember particular historical events, others not. It might be that dark ages after ancient Greece could be interpreted that way - by some reason total majority of historians do not remember events of that time.

(iii) Let us consider a set of agents: admins, - analyzing behavior of a network. Sequences of their steps in this analysis may be paths along the same transmission line. But paths may be different - some may have gaps because some admins could have no authority, passwords to achieve protocols located in particular nodes of the network. So these nodes will be not achievable for their analysis.

In the sequel, we may consider any state $a \in N$ as a time point and any $[\xi, d(\xi)]$ as the time interval available for the agents responsible for verification/reasoning in the time points $a \in [\xi, d(\xi)]$. Time ($\leq, R_{\xi,j}$ – in our formalization) may be directed to past or future - depending on the particular case which we wish to model. Now we would like to comment the choice of our models in a bit more formal manner.

(1) Interpretation of $R_{\xi,j}$ as agent's accessibility relations. We consider relations $R_{\xi,j}$ on $[\xi, d(\xi)]$ with domains which do not compulsory include all worlds from $[\xi, d(\xi)]$. That is admitted to represent some cases when certain elements of past may be forgotten as it is commented above, when the information is incomplete.

(2) All the intervals of time $[\xi, d(\xi)], \xi \in In$ are finite to represent bounded amount of memory about past. Though links from $[\xi, d(\xi)]$ to $[d(\xi), d(d(\xi))]$ via $Nxt(d(\xi)) = d(\xi) + 1$ and all $R_{d(\xi),j}$ are present; so, by intransitive iterations we may go to past and analyze collected information. In particular, we obtained that all R_j may be linear but not transitive.

(3) The operations U_j are defined also in non-standard way in order to represent forgettable past (lacunas) in memory:

$$(\mathcal{ALF}_{In}, a) \Vdash_V (\varphi\ U_j\ \psi) \ \Leftrightarrow\ \exists \xi(a \in It(\xi))\ \&\ \exists b[(aR_{\xi,j}b) \wedge ((\mathcal{ALF}_{In}, b) \Vdash_V \psi) \wedge$$

$$\forall c[(c \in Dom_{\xi,j}\ \&\ aR_{\xi,j}c\ \&\ c < b) \Rightarrow (\mathcal{ALF}_{In}, c) \Vdash_V \varphi]].$$

This rule says that there is a state b in past before a, which the agent j remember, where the statement ψ is true, and in all states which are situated after b but before a and *which the agent j still remember* the statement φ is true.

(4) The presence of different agent's accessibility relations $R_{\xi,j}$ does not allow simply ignore states which are outside the domain of a particular relation $R_{\xi,j}$, but inside $[\xi, d(\xi)]$ because these states may be inside domains of some other relations R_{ξ,j_1}. So, $\varphi\ U_1\ \psi$ may be true at some state a, but $\varphi\ U_2\ \psi$ to be false at a.

Examples.

(i) The formula $\neg\Diamond_1\varphi \wedge N\varphi$ represents impossible for the standard linear temporal logics property: The statement φ is true at the next temporal point, but still is impossible from temporal viewpoint. In our agent's interpretation we have: φ was true at yesterday, but the agent 1 does not remember yesterday.

(ii) Else unusual property: $\neg\Diamond_1\varphi \wedge N\Box_1\varphi$. This formula says that yesterday and always before it the statement φ was true at all states which the agent 1 remember. But today the agent 1 does not remember that yesterday and what always been before yesterday (that is there is a lacuna in his memory since yesterday).

(iii) The formula $\Box_1\varphi \wedge \Diamond_1[N(\neg\varphi \wedge \Box_1\varphi)]$ describes the following environment. Since today and until the last remembered state the statement φ was true from viewpoint of the agent a_1 in the all states which he remember (knows). But at the last remembered state φ was false at its predecessor, though again true in all its remembered predecessors.

(iv) The formula $(\varphi U_1\psi) \wedge (\neg\psi) \wedge \neg\Diamond_1\varphi$ says that not today, but in past ψ was true, but the agent a_1 does not remember it, in particular, no one state before the first one where ψ was true.

(v) The formula $\varphi \wedge (\varphi \rightarrow N\varphi) \wedge \Box_1(\varphi \rightarrow N\varphi) \wedge \neg NN\varphi$ says that today and yesterday φ was true, but the agent a_1 does not remember yesterday.

(vi) The formula $\Box_1\varphi\wedge\varphi\wedge\neg\Box_1\Box_1\varphi$ says, in particular, that the time is not transitive from viewpoint of the agent (a1). But actually, this formula says bigger that this. It says that the agent (a1) may achieve the last remembered state, and only being in this state (a1) may achieve one still later sate where φ was false.

(v) The formula $[\bigvee_{X\subseteq Ag,||X||\leq||Ag||/2}[\bigwedge_{j\in X}(\varphi U_j\psi)]] \rightarrow \varphi U_1\psi$ says that the agent (a1) very easy follows the majority of the agents Ag: if at least half of agents believe that φ holds until ϕ will hold, (a1) also accept it.

3 Decision Algorithm

Recall that for any logic L the satisfiability problem is to determine by any given formula φ if this formula is satisfiable in L. That is, if there is a model where φ true (if there is an algorithm answering this question for any given φ, the satisfiability problem is said to be decidable). A logic L is decidable if there is an algorithm answering for any formula φ if $\varphi \in L$ holds (if φ is a theorem of L). These problems are mutually connected: φ is satisfiable in L iff $\neg\varphi \notin L$; $\varphi \in L$ iff $\neg\varphi$ is not satisfiable; so decidability itself implies that satisfiability problem is decidable and vice versa.

Definition 4 *A formula φ is said to be* refuted *at a world $a \in W_{\mathcal{ALF}_{In}}$ of a model based at a frame $\mathcal{ALF}_{In}$ with a valuation V iff $(\mathcal{ALF}_{In}, a) \not\Vdash_V \varphi$.*

For any model based at a frame

$$\mathcal{ALF}_{In} = \langle W_{\mathcal{ALF}_{In}}, (\bigcup_{\xi\in In, j\in[1,k]} \langle R_{\xi,j}\rangle), Nxt, \rangle$$

and a natural number $m \geq 0$ the model $\mathcal{ALF}_{In}(m)$ obtained from $\mathcal{ALF}_{In}$ is the model based on the set

$$|\mathcal{ALF}_{In}| \setminus \bigcup\{It[\xi] \mid \xi \in In, \xi = d^t(0),\ t > m+2\},$$

where we re-define Nxt assuming $Nxt(d^{m+2}(0)) = d^{m+2}(0)$ and any accessibility relation $R_{d^{m+2}(0),j}$ to be the standard $\leq$ on $[d^{m+2}(0), d^{m+2}(0)]$, and assume all letters to be true at $d^{m+2}(0)$. The definition of inductive steps for computation of the truth values of formulas remains to be the same as earlier. For a formula α, $td(\alpha)$ denotes the temporal degree of α.

Lemma 5 *Let α be a formula and $td(\alpha) = g$. If α is refuted in a frame $\mathcal{ALF}_{In} = \langle W_{\mathcal{ALF}_{In}}, (\bigcup_{\xi \in In, j\in[1,k]} \langle R_{\xi,j} \rangle), Nxt, \rangle$ at the world 0 then α can be refuted at 0 in a model based on a frame $\mathcal{ALF}_{In}(g+1)$.*

Proof is a rather standard induction. We need an axillary statement: for any $m \in N$, for all formulas β,

$$td(\beta) \leq n \;\Rightarrow [\forall a \in [d^m(0), d^{m+1}(0)] \subseteq \mathcal{ALF}_{In}$$

$$(\mathcal{ALF}_{In}, a) \Vdash_V \beta \Leftrightarrow (\mathcal{ALF}_{In}(m+n+r) \Vdash \beta, \forall r \geq 1. \tag{1}$$

Indeed, for $n = 0$ it is evident. Let (1) holds for all $n \leq n_1$ and we have a formula β of temporal degree n_1+1. Then β is a formula constructed out of formulas β_i with temporal degree at most n_1 and some formulas γ_i with temporal degree n_1+1 by boolean logical operations. For formulas β_i we apply (1) to the interval $[d^m(0), d^{m+1}(0)]$ itself and obtain:

$$\forall a \in [d^m(0), d^{m+1}(0)],\; (\mathcal{ALF}_{In}, a) \Vdash_V \beta_i \Leftrightarrow$$

$$(\mathcal{ALF}_{In}(m+n_1+r), a) \Vdash \beta_i, \forall r \geq 1 \tag{2}$$

For any formula γ_i we may assume, $\gamma_i = \delta_{i,1} U_j \delta_{i,2}$, or $\gamma_i = N\alpha_i$. If $\gamma_i = \delta_{i,1} U_j \delta_{i,2}$ by (2) we obtain:

$$\forall a \in [d^m(0), d^{m+1}(0)],\; (\mathcal{ALF}_{In}, a) \Vdash_V \delta_{i,k} \Leftrightarrow$$

$$(\mathcal{ALF}_{In}(m+n_1+r), a) \Vdash \delta_{i,k}, \forall r \geq 1, \tag{3}$$

and hence $\forall a \in [d^m(0), d^{m+1}(0)]$

$$(\mathcal{ALF}_{In}, a) \Vdash_V \delta_{i,1} U_j \delta_{i,2} \Leftrightarrow (\mathcal{ALF}_{In}(m+n_1+r), a) \Vdash \delta_{i,1} U_j \delta_{i,2},\; \forall r \geq 1.$$

Let $\gamma_i = N\alpha_i$. If $a < d^{m+1}(0)$ we apply the same reasoning as for $\gamma_i = \delta_{i,1} U_j \delta_{i,2}$ above. If $a = d^{m+1}(0)$ we first apply (2) to the interval $[d^{m+1}(0), d^{m+2}(0)]$ and obtain:

$$\forall b \in [d^{m+1}(0), d^{m+2}(0)],\; (\mathcal{ALF}_{In}, b) \Vdash_V \alpha_i \Leftrightarrow$$

$$(\mathcal{ALF}_{In}(m+n_1+r), b) \Vdash \alpha_i,\; \forall r \geq 1.$$

Consequently,

$$(\mathcal{ALF}_{In}, d^{m+1}(0)) \Vdash_V N\alpha_i \Leftrightarrow (\mathcal{ALF}_{In}(m+n_1+r), d^{m+1}(0) \Vdash_V N\alpha_i.$$

Thus, we showed that (1) holds for $n = m_1 + 1$, and by induction it holds for all n. This conduces the proof of our lemma if we take $m = 0$. Q.E.D.

Lemma 6 *If a formula α is refuted in a model based at a frame $\mathcal{ALF}_{In}(m)$ at the world* 0*, then α may be refuted in a model based at a standard frame $\mathcal{ALF}_{In}$.*

Proof is trivial: it is sufficient to blow out the frame $\mathcal{ALF}_{In}(m)$ to the infinite one. Indeed, consider the final interval $[d^m(0)), d^{m+1}(0)]$ of the frame $\mathcal{ALF}_{In}(m)$ and adjoin to $\mathcal{ALF}_{In}(m)$ starting from $[d^{m+1}(0), d^{m+1}(0)]$ infinite sequence of intervals $[a_i, b_i], i \in N$ with $Nxt(a_i) = b_i$, $Nxt(b_i) = a_{i+1}$ and the valuation of the letters on this new worlds to be the same as it has been at the state $d^{m+1}(0)$. It is easy to see that this modification will not effect the truth values of formulas on the initial interval $[0, d^{m+1}(0)]$. Q.E.D.

Therefore, due to Lemmas 5 and 6, bearing in mind to find solving algorithm for decidability we may restrict ourselves with only models based on frames of kind $\mathcal{ALF}_{In}(m)$.

For our logic MA^{Int}_{Lin} the usual standard techniques (as for instance in [19, 18]), cannot be directly implemented because the time relations are not transitive. For instance, the non-transitivity hampers to convert formulas in more suitable and simple forms. The technique of reduction formulas to rules (which we already used earlier many times for different purpose (cf. e.g.. [17, 19, 23]) here will be very useful. This approach efficiently simplifies all proofs because it allows to consider very simple and uniform formulas without nested temporal operations. Few definitions below will explain this approach. A rule is an expression

$$\mathbf{r} := \frac{\varphi_1(x_1, \dots, x_n), \dots, \varphi_l(x_1, \dots, x_n)}{\psi(x_1, \dots, x_n)},$$

where $\varphi_1(x_1, \dots, x_n), \dots, \varphi_l(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are formulas constructed out of letters (variables) $x_1, \dots, x_n$.

Meaning of $\mathbf{r}$ is: $\psi(x_1, \dots, x_n)$ (which is called conclusion) follows (logically follows) from assumptions $\varphi_1(x_1, \dots, x_n), \dots, \varphi_l(x_1, \dots, x_n)$. The definition of a rule to be valid is the same for any relational model:

Definition 7 *For a model $\mathcal{M}$, $\mathbf{r}$ is* valid (true) *in $\mathcal{M}$ iff:* $[\forall a\, ((\mathcal{M}, a) \Vdash_V \bigwedge_{1\leq i\leq l} \varphi_i)] \;\Rightarrow\; \forall a\, ((\mathcal{M}, a) \Vdash_V \psi)$*. That is: as soon as all premises of* $\mathbf{r}$ *are true at all states from $\mathcal{M}$ then the conclusion if true at all states as well. If this is not a case, we say* $\mathbf{r}$ *is refuted in $\mathcal{M}$, or refuted in $\mathcal{M}$ by V, and write $\mathcal{M} \nVdash_V \mathbf{r}$. A rule $\mathbf{r}$ is valid in a frame $\mathcal{F}$ (notation $\mathcal{F} \Vdash \mathbf{r}$) if it is valid in any model based at $\mathcal{F}$.*

For any formula φ, we may convert it to rule rule $x \to x/\varphi$.

Lemma 8 *For any formula φ, φ is a theorem of MA^{Int}_{Lin} (that is $\varphi \in MA^{Int}_{Lin}$) iff the rule $(x \to x/\varphi)$ is valid in any frame for MA^{Int}_{Lin}.*

Proof is self evident.

Definition 9 *A rule* $\mathbf{r}$ *is said to be in* reduced normal form *if* $\mathbf{r} = \varepsilon/x_1$ *where*

$$\varepsilon := \bigvee_{1\leq j\leq l} [\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (Nx_i)^{t(j,i,1)} \wedge \bigwedge_{l\in[1,k],1\leq i,k\leq n} (x_i\, U_l x_k)^{t(j,i,k,l)}],$$

$t(j,i,z), t(j,i,k,l) \in \{0,1\}$ *and, for any formula* α *above,* $\alpha^0 := \alpha$, $\alpha^1 := \neg\alpha$.

Definition 10 *For any given rule* $\mathbf{r}$*, a rule* $\mathbf{r_{nf}}$ *in the reduced normal form is a normal form of* $\mathbf{r}$ *iff, for any frame* $\mathcal{ALF}_{In}$ *for our logic* MA^{Int}_{Lin} *the following holds* $\mathcal{ALF}_{In} \Vdash \mathbf{r} \Leftrightarrow \mathcal{ALF}_{In} \Vdash \mathbf{r_{nf}}$.

Theorem 11 *There exists an algorithm running in (single) exponential time, which, for any given rule* $\mathbf{r}$*, constructs some its reduced form* $\mathbf{r_{nf}}$.

Proof of similar statement for various logics been suggested by us quite a while ago (e.g. it is a literal repetition of the proof for the logic $\mathcal{LTL}$ itself; for instance, cf. Lemma 5 in [2], or proof of similar statements in [16, 17]). Q.E.D.

Bearing in mind suggested sequence of steps, a formula φ is a theorem of MA^{Int}_{Lin} (i.e. $\varphi \in MA^{Int}_{Lin}$) iff the rule $r := p \to p/\varphi$ is valid at all frames $\mathcal{ALF}_{In}$ (cf. Lemma 8) and iff its reduced form $\mathbf{r_{nf}}$ is valid at all frames $\mathcal{ALF}_{In}$ (cf. Theorem 11). This, to solve the satisfiability problem, we may consider only rules in reduced forms.

Lemma 12 *If a rule in a normal form* $\mathbf{r_{nf}}$ *is refuted in a model based at a frame* $\mathcal{ALF}_{In}(g)$ *then* $\mathbf{r_{nf}}$ *can be refuted in some such frame where* $\forall\xi \in In$, $d(\xi) - \xi \leq r(D) + 2$, *where* $r(D)$ *is the number of disjuncts in* $\mathbf{r_{nf}}$.

Proof. Let $\mathbf{r_{nf}} = \varepsilon/x_1$ where $\varepsilon = \bigvee_{1\leq j\leq v} \theta_j$,

$$\theta_j = [\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (N\ x_i)^{t(j,i,1)} \wedge \bigwedge_{s\in[1,k],1\leq i,k\leq n} (x_i\, U_s x_k)^{t(j,i,k,l)}];$$ let

$\mathbf{r_{nf}}$ be refuted in a given frame $\mathcal{ALF}_{In}(g)$ by a valuation V.

By our assumption about the refutation the rule, for any a from the base set of $\mathcal{ALF}_{In}(g)$ there is some unique disjunct θ_j from the premise of $\mathbf{r_{nf}}$ which is true at a:

$$(\mathcal{ALF}_{In}(g), a) \Vdash_V \theta_j;$$

denote this unique disjunct by $\theta(a)$.

Consider any $[\xi, d(\xi)]$ for $\xi \in In$. If $d(\xi) = \xi + 1$ we do nothing. Otherwise consider $Nxt(\xi)$ and the greatest number $Nx_1(\xi)$ from $[\xi, d(\xi))$ strictly bigger than $Nxt(\xi)$ (if one exists) such that $\theta(Next(\xi)) = \theta(Nx_1(\xi))$.

We now rarefy $[\xi, d(\xi)]$ by removing all states situated strictly between ξ and $Nx_1(\xi)$ and set that $Nxt(\xi) = Nx_1(\xi)$, the valuation on the model remains to be the same as earlier. The relatuions R$DenotetheobtainingframebyALF_{In}(r,g)$ and the model by $\mathcal{MALF}_{In}(r,g)$.

Lemma 13 *For any* $a \in [\xi, d(\xi)] \cap \mathcal{ALF}_{In}(r,g)$ *and any* θ_δ,

$$(\mathcal{ALF}_{In}(g), a) \Vdash_V \theta_\delta \Leftrightarrow (\mathcal{ALF}_{In}(r,g), a) \Vdash_V \theta_\delta. \quad (4)$$

Proof. If $a \geq Nx_1(\xi)$ this is evident because we changed nothing above $Nx_1(\xi)$. If $a = \xi$ then the correspondences of the truth of the components of θ_j are evident for all the cases except the operations U_s.

Assume first that $(\mathcal{ALF}_{In}(g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$. Then there is $b \in Dom_{\xi,s}$ such that

$$[(\xi R_{\xi,s} b) \wedge ((\mathcal{ALF}_{In}(g), b) \Vdash_V x_{i,2}) \,\&$$

$$\forall c[(c \in Dom_{\xi,s} \,\&\, \xi R_{\xi,s} c \,\&\, c < b) \Rightarrow (\mathcal{ALF}_{In}(g), c) \Vdash_V x_{i_1}]].$$

Take minimal b with these properties. If $b = \xi$ all is clear. If $b > \xi$ then $b \geq Nxt(\xi)$ and if $b \geq Nx_1(\xi)$ then we have

$$(\mathcal{ALF}_{In}(g), Nx_1(\xi)) \Vdash_V x_{i_1} U_s x_{i,2},$$

and consequently $(\mathcal{ALF}_{In}(r, g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$.

Assume now that $b < Nx_1(\xi)$ and $b > \xi$, that is $Nxt(\xi) \leq b < Nx_1(\xi)$. Then because $\theta(Nxt(\xi)) = \theta(Nx_1(\xi))$ we have

$$(\mathcal{ALF}_{In}(g), Nx_1(i)) \Vdash_V x_{i_1} U_s x_{i,2} \quad \text{and } (\mathcal{ALF}_{In}(r, g), Nx_1(i)) \Vdash_V x_{i_1} U_s x_{i,2}$$

and since $(\mathcal{ALF}_{In}(g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$ we conclude that if $\xi \in Dom_{\xi,s}$ than $(\mathcal{ALF}_{In}(d), \xi) \Vdash_V x_{i_1}$, and consequently

$$(\mathcal{ALF}_{In}(r, g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}.$$

For opposite direction, let $(\mathcal{ALF}_{In}(r, g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$. Than

$$\exists b \in |\mathcal{ALF}_{In}(r, g)|, \ \ [(\xi R_{\xi,s} b) \wedge (\mathcal{ALF}_{In}(r, g), b) \Vdash_V x_{i,2}) \,\&$$

$$\forall c[(c \in \mathcal{ALF}_{In}(r, g) \cap Dom_{\xi,s} \,\&\, \xi R_{\xi,s} c \,\&\, c < b) \Rightarrow (\mathcal{ALF}_{In}(r, g), c) \Vdash_V x_{i_1}].$$

Take minimal b with this property. If $b = \xi$ then all is done. Otherwise $b \geq Nx_1(\xi)$ and consequently

$$(\mathcal{ALF}_{In}(r, g), Nx_1(\xi)) \Vdash_V x_{i_1} U_s x_{i,2} \text{ and } (\mathcal{ALF}_{In}(g), Nx_1(\xi)) \Vdash_V x_{i_1} U_s x_{i,2}$$

Since $\theta(Nxt(\xi)) = \theta(Nx_1(\xi))$ we conclude that

$$(\mathcal{ALF}_{In}(g), Nxt(\xi)) \Vdash_V x_{i_1} U_s x_{i,2}.$$

If $\xi \notin Dom_{\xi,s}$ we obtain $(\mathcal{ALF}_{In}(g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$.

If $\xi \in Dom_{\xi,s}$, then $(\mathcal{ALF}_{In}(r, g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$ implies

$$(\mathcal{ALF}_{In}(r, g), \xi) \Vdash_V x_{i_1}$$

and $(\mathcal{ALF}_{In}(g), \xi) \Vdash_V x_{i_1}$ what together with $\mathcal{ALF}_{In}(g), Nxt(\xi)) \Vdash_V x_{i_1} U_s x_{i,2}$ allows us to conclude that $(\mathcal{ALF}_{In}(g), \xi) \Vdash_V x_{i_1} U_s x_{i,2}$. Lemma 13 is proved. Q.E.D.

Now, instead of $[\xi, d(\xi)]$ we consider the interval $[Nx_1(\xi)(d(\xi))]$ and apply to $[Nx_1(\xi), d(\xi))]$ the same rarefication procedure as for $[\xi, d(\xi)]$ above. Analog of

Lemma 13 will hold, and it will not effect the truth of formulas θ_δ on ξ. We then continue this procedure which stops in at most $r(D)$ steps and we obtain $d(\xi) - \xi \leq r(D) + 2$. Doing this transformation for all $\xi \in In$ and $[\xi, d(\xi)]$ we accomplish the proof of Lemma 12. Q.E.D.

Combining Lemmas 5, 6, 8, 12 and Theorem 11 we obtain

Theorem 14 *The satisfiability problem for the logic MA^{Int}_{Lin} is decidable: there is an algorithm, described in the series of cited lemmas, which verify satisfiability. The logic MA^{Int}_{Lin} is decidable.*

We would like to conclude this section by extensions obtained results to rules, but not only to formulas themselves. As we demonstrated, rules are more general formalization and they do cover the case of formulas by consideration rules with always true premise.

Definition 15 *A rule $\mathbf{r}$ is valid in logic MA^{Int}_{Lin} if for any frame $\mathcal{ALF}_{In}$ from the set of frames generating MA^{Int}_{Lin}, $\mathbf{r}$ is valid at MA^{Int}_{Lin}.*

Similar to the Theorem 14 above, using the same preliminary results, we obtain

Theorem 16 *Logic MA^{Int}_{Lin} is decidable w.r.t. valid inference rules: there is an algorithm, described in the series of cited lemmas, which verify validness.*

We pause briefly to comment this result. The validness of a rule in any transitive temporal (for instance in LTL itself) or modal logic may be easily represented by formula. E.g. a rule φ/ψ is valid iff $\Box\varphi \wedge \varphi \to \psi$ is a theorem of LTL (where as earlier $\Diamond\varphi := \top U\varphi, \Box := \neg\Diamond\neg$). It is not a case for intransitive temporal or modal logics because any operation $\Box$ and even any finite composition of it $\Box^n$ are always local and cannot describe properties of all frames. For instance, for any numbers k and n the rule $N^k\Box^n x/x$ is valid in our logic MA^{Int}_{Lin} (it is an easy exercise to verify it), but no way to describe it by a formula.

4 Transitivization, Insight to all Past

In this concluding section, using collected techniques we would like to study a version of the MA^{Int}_{Lin}, when we will admit infinite insight, an agent which may check all past, remembers all past. This looks like we would like to admit logical operations of sort: $\Diamond^\infty\varphi = \exists m\Diamond^m\varphi$, so to say - transitivization of our agent's accessibility relations. Or even more precisely, to admit the operation: $\Box^\infty\varphi = \varphi \wedge \forall m N^m\varphi$. This resembles a need for a technique similar to the used one for conception of common knowledge from [5]. We would like to approach this task via solution more general case: to consider standard $\leq$ on N as a new additional relation on $\mathcal{ALF}_{In}$ - it is the standard linear order, and consider the new logical operation U - simply the standard until operation as it has been defined at the linear temporal logic itself initially. That is we will interpret U as follows:

$$\forall a \in N, (\mathcal{ALF}_{In}, a) \Vdash_V (\varphi U\psi) \Leftrightarrow \exists b[(a \leq b) \wedge ((\mathcal{ALF}_{In}, b) \Vdash_V \psi)\wedge$$

$$\forall c[(a \le c < b) \Rightarrow (\mathcal{ALF}_{In}, c) \Vdash_V \varphi]].$$

The modified frame will be denoted as $\mathcal{ALF}_{In,\le}$, for sake of uniform notation, in the sequel, U will be denoted by U_0.

Definition 17 *The logic $MA_{Lin}^{Int,\infty}$ is the set of all formulas which are valid in any model based at any $\mathcal{ALF}_{In,\le}$ frame.*

To work with this logic we may directly use results and elements of proofs from the previous section which will make proofs shorter, transparent and very compact.

Lemma 18 *If a rule in a normal form $\mathbf{r_{nf}}$ is refuted in a model based at a frame $\mathcal{ALF}_{In,\le}$ then $\mathbf{r_{nf}}$ can be refuted in some such frame where $\forall \xi \in In$, $d(\xi) - \xi \le r(D) + 2$, where $r(D)$ is the number of disjuncts in $\mathbf{r_{nf}}$.*

Proof. We may verbatim repeat the proof of Lemma 12. The matter is that the presence of U_0 does not effect the steps of proof for (4). Simply we consider U_0 as one of U_s inside intervals $[\xi, d(\xi)]$ and we do the steps for the proof of (4) for all ξ simultaneously. Q.E.D.

For any frame $\mathcal{ALF}_{In,\le}$ (or a model based on a such frame) and any numbers $m = d^v(0), v > 2, n = d^{v_1}(0) > d^{v+1}(0)$ from the base set of this frame (similar to the earlier), $\mathcal{ALF}_{In,\le,m,n}$ is the frame (model) obtained from the original one by assumption $Nxt(d^{v_1-1}(0)) = d^v(0)$ and deleting totally all numbers bigger than $d^v(0)$.

Lemma 19 *If a rule in reduced normal form $\mathbf{r_{nf}}$ is refuted in model based at a frame $\mathcal{ALF}_{In,\le}$ at 0 with $\forall \xi \in In$, $d(\xi) - \xi \le r(D) + 2$, then $\mathbf{r_{nf}}$ can be refuted in some frame $\mathcal{ALF}_{In,\le,m,n}$, for $m = d^v(0), v > 2, n = d^{v_1}(0) > d^{v+1}(0)$ at 0, with $\forall \xi \in In$, $d(\xi) - \xi \le r(D) + 2$.*

Proof. For any $x_{i,1} U_s x_{i_2}$, where $(\mathcal{ALF}_{In,\le}, a) \Vdash_V x_{i,1} U_s x_{i_2}$ the realizers for x_{i_2} are all worlds $b \in \mathcal{ALF}_{In,\le}$ such that

(i) $(aR_{\xi,s}b) \wedge ((\mathcal{ALF}_{In,\le}, b) \Vdash_V x_2)$ and

(ii) $\forall c[(c \in Dom_{\xi,s} \,\&\, aR_{\xi,s}c \,\&\, c < b) \Rightarrow (\mathcal{ALF}_{In,\le}, c) \Vdash_V x_1]$.

For any $x_{i,1} U_s x_{i_2}$, where $(\mathcal{ALF}_{In,\le}, d^2(0)) \Vdash_V x_{i,1} U_s x_{i_2}$ where realizers for x_{i_2} are situated strictly above $d^2(0)$ consider smallest realizers $r(x_{i,1} U_s x_{i_2})$ from $\mathcal{ALF}_{In,\le}$, they form a set S. Let the maximal among them be the number n_m.

From assumption about the refutation the rule, it follows that for any a from the base set of our frame, there is some unique disjunct θ_j from the premise of $\mathbf{r_{nf}}$ which is true at a:

$$(\mathcal{ALF}_{In,\le}, a) \Vdash_V \theta_j;$$

denote this unique disjunct by $\theta(a)$.

Consider the minimal number $m_n > n_m$, for which for any $d^v(0) > m_n$ the list

$$[\theta(d^v(0)), \ldots, \theta(d^{v+1}(0))]$$

repeats for other $v_1 > v$ infinitely many times.

Consider the first interval $[d^v(0), d^{v+1}(0)]$ strictly above m_n and the set S_r of all smallest realizers for all possible $x_{i,1} U_s x_{i_2}$ which are true at $d^{v+1}(0)$. Let m_1 be any number strictly bigger than any one in S_r.

Consider the earliest interval $[d^{v+w}(0), d^{v+w+1}(0)]$ situated strictly above the initial interval $[d^v(0), d^{v+1}(0)]$ where $d^{v+w}(0) > m_1$ and the list

$$[\theta(d^{v+w}(0)), \ldots, \theta(d^{v+w+1}(0))]$$

repeats the list $[\theta(d^v(0)), \ldots, \theta(d^{v+1}(0))]$. We modify now the frame by direction: $Nxt(d^{v+w}(0) - 1) = d^v(0)$ and then we delete all worlds situated strictly above $d^{v+w}(0) - 1$.

Lemma 20 *This transformation remains the truth of all subformulas of the rule* $\mathbf{r_{nf}}$ *to be the same as it was in the original model.*

Proof. It is immediate computation bearing in mind the presence of the set of realizers S_r chosen above being situated before $[d^{v+w}(0), d^{v+w+1}(0)]$. Q.E.D.

Using this statement we complete the proof for our Lemma 19.

Lemma 21 *If a rule in normal form* $\mathbf{r_{nf}}$ *is refuted in a model based at a frame* $\mathcal{ALF}_{In,\leq,m,n}$, *for* $m = d^v(0), v > 2, n = d^{v_1}(0) > d^{v+1}(0)$ *at* 0 *with* $\forall \xi \in In, d(\xi) - \xi \leq r(D) + 2$, *then* $\mathbf{r_{nf}}$ *can be refuted in some frame* $\mathcal{ALF}_{In,\leq,m_1,n_1}$ *of size effectively computed from the size of* $\mathbf{r_{nf}}$.

Proof. Indeed, first we, moving from $d^2(0)$ upwards, delete all intervals $[d^w(0), \ldots, d^{w+1}(0)]$ situated before $m = d^v(0) > 2$ with repeated

$$[\theta(d^w(0)), \ldots, \theta(d^{w+1}(0))]$$

together with worlds situated between of them.

Lemma 22 *The deletion of any such repetition* $[d^w(0), \ldots, d^{w+1}(0)]$ *does not change the truth values of sub-formulas of the rule* $\mathbf{r_{nf}}$.

Proof is an immediate computation based on the structure of the frame and the fact of the repetition. Q.E.D.

After this transformation the part of the frame situated before $d^v(0)$ is not only finite but else it has the effectively computable size.

Now we pick up all minimal from $d^v(0)$ intervals $[d^f(0), d^{f+1}(0)]$ situated after $d^v(0)$ (that is situated in the circle part) and containing minimal realizers $r(x_{i,1} U_s x_{i_2})$ for $x_{i,1} U_s x_{i_2}$ which are true at $d^v(0)$.

Let X_r be the set of all these intervals. Now we repeat the procedure described above for the interval $[0, d^v(0)]$ but subsequently, strictly inside between of each two neighboring different intervals $I_1, I_2 \in X$. We delete repetitions of the intervals situated strictly between them, moving from the end of I_1 to the beginning of I_2. And then we move along the circle repeating this transformation for each pair $I_1, I_2 \in S$. This transformation will not affect the truth of sub-formulas of $\mathbf{r_{nf}}$ thanks to our choice of X_r. Lemma 21 is proved. Q.E.D.

Lemma 23 *If a rule in normal form* $\mathbf{r_{nf}}$ *is refuted in some frame* $\mathcal{ALF}_{In,\leq,m,n}$ *then* $\mathbf{r_{nf}}$ *may be refuted in a frame* $\mathcal{ALF}_{In,\leq}$.

Proof is an easy observation that if we will stretch the circle part from the frame $\mathcal{ALF}_{In,\leq,m,n}$ rolling the circle part towards future in the standard manner, we will obtain the frame $\mathcal{ALF}_{In,\leq}$ which preserves the truth values of sub-formulas of $\mathbf{r_{nf}}$ at the worlds and theirs copies. Q.E.D.

Using analog of Theorem 11, analog of Lemma 8, Lemmas 18, 19, 21 and 23 we obtain

Theorem 24 *The satisfiability problem for the logic* $MA_{Lin}^{Int,\infty}$ *is decidable: there is an algorithm, described in the series of cited lemmas, which verify satisfiability. The logic* $MA_{Lin}^{Int,\infty}$ *is decidable.*

5 Open problems

There are many remaining open problems in suggested framework. We first would point important typical open yet problems: the problems of axiomatization for both logics MA_{Lin}^{Int} and $MA_{Lin}^{Int,\infty}$, the problems of recognizing admissible rules in these logics. The question of extension this framework to temporal logics where agent's accessibility relations are not compulsory restricted to intervals $[\xi, d(\xi)]$ but may arbitrary reach upper divisions is not investigated yet (we do not see yet now how to overcome appearing technical difficulties; even the problem of satisfiability for such logics is open yet). Next open attractive direction is to consider modification of suggested relational non-transitive models to the case of not-discrete but continues models, based e.g. at rational, or all real numbers.

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