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One Proof of the Fundamental Theorem of Algebra (of Polynomials)

Olga V. Khodos*

Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

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One proof of the fundamental theorem of algebra (of polynomials) is given, based on the theorem on the number of zeros of an entire function.

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It is well known how important was Gauss's proof of the fundamental theorem of algebra. Since then many other proofs have been found, see, for example, the classical textbooks on algebra by Kurosh [1], van der Waerden [2] etc. Here we give a proof based on recent results related to zeros of entire functions [3].

Suppose that a function $f = f(z)$ of the complex variable z is an entire function of finite order of growth ρ and has the form

$$f(z) = \sum_{k=0}^{\infty} b_k \cdot z^k, \quad f(0) = b_0 = 1. \quad (1)$$

Recall that an entire function $f(z)$ has a finite order (of growth) if there exists a positive number A such that

$$f(z) = O(e^{R^A}) \quad \text{for } |z| = R \rightarrow +\infty.$$

The infimum of such A is called the *order* of the function f .

Note that if $b_0 = 0$, then the function must have a zero. Denote by γ_r the circle

$$\gamma_r = \{z : |z| = r\}, \quad r > 0.$$

We have the following theorem [3]:

Theorem 1. *The function f has no zeros in $\mathbb{C}$ if and only if for sufficiently small r*

$$\Delta_k = \int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = 0$$

for all $k > \rho$.

This theorem can be reformulated as follows [3]:

Theorem 2. *For an entire function f to have no zeros in $\mathbb{C}$, it is necessary and sufficient that all determinants*

*KhodosOlga@mail.ru

$$\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ 3b_3 & b_2 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix}$$

equal to zero for k strictly greater than ρ .

In fact, these determinants are equal to the integrals Δ_k [3].

As a consequence, we obtain the condition that an entire function has zeros in $\mathbb{C}$.

Corollary 1. *For an entire function f to have zeros in $\mathbb{C}$ it is necessary and sufficient that $\Delta_k \neq 0$ for some $k > \rho$.*

Now let $f(z)$ be a polynomial $P_n(z)$ of degree $n > 0$

$$P_n(z) = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n,$$

$$b_0 = P_n(0) = 1.$$

It is well known that a polynomial is an entire function of zero order. Consider the system $\Delta_1, \dots, \Delta_n$ (n is the degree of the polynomial). We shall show that in this system at least one determinant is not equal to 0.

Suppose that they all are equal to 0. Consider Δ_1 , which is equal to b_1 , so $b_1 = 0$.

Consider now Δ_2 :

$$\Delta_2 = \begin{vmatrix} b_1 & b_0 \\ 2b_2 & b_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2b_2 & 0 \end{vmatrix} = -2b_2 = 0$$

and therefore $b_2 = 0$.

Let $b_1 = 0, \dots, b_{k-1} = 0$ for some $k \leq n$. We show that $b_k = 0$.

Write out the determinant Δ_k :

$$\Delta_k = \begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ 3b_3 & b_2 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ kb_k & 0 & 0 & \dots & 0 \end{vmatrix} = (-1)^{k+1} k b_k I_{k-1},$$

where I_{k-1} is the identity matrix of order $k-1$.

$$\Delta_k = (-1)^{k+1} k b_k I_{k-1} = 0,$$

hence $b_k = 0$.

Therefore, by the principle of complete mathematical induction, $b_1 = b_2 = \dots = b_n = 0$. That is, the polynomial P_n is equal to 1 and has degree 0. We arrive at a contradiction.

Therefore, in the system $\Delta_1, \Delta_2, \dots, \Delta_n$, at least one determinant is not equal to 0. Then by the corollary 1, the polynomial $P_n(z)$, $n > 0$, has zeros in $\mathbb{C}$. This is another proof of the fundamental theorem of algebra (of polynomials).

Let a function $f(z)$ of the form (1) be an entire function of order $\rho < 1$.

Theorem 3. *A function $f(z)$ of the form (1) has zeros in $\mathbb{C}$.*

Proof. Since $\rho < 1$, then we must consider determinants $\Delta_1, \Delta_2, \dots, \Delta_k, \dots$, and show that one of them does not equal to 0.

Suppose that $\Delta_1 = \Delta_2 = \dots = \Delta_k = \dots = 0$. Then doing the same steps as above we see that all coefficients $b_1 = b_2 = \dots = b_k = \dots = 0$. Therefore, $f(z)$ is a constant and is not an entire transcendental function (which must have infinitely many terms with coefficients different from zero). $\square$

Corollary 2. *An entire transcendental function of order $\rho < 1$ has infinite number of zeros.*

Proof. Let $z = a$ be a zero of such a function. Dividing the function by $z - a$ we get a function of the same order $\rho < 1$, which by the previous theorem has zeros. $\square$

If ρ is fractional, then Theorem 2 is well-known [4, Sec. 8], [5].

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References

- [1] A.G.Kurosh, Higher Algebra, Moscow, Mir, 1975.
- [2] B.L.Van der Waerden, Algebra, Springer-Verlag, Berlin-Heidelberg-New-York, 1971.
- [3] A.M.Kytmanov, O.V.Khodos, On localization of zeros of an entire function of finite order of growth, *Journal Complex Analysis and Operator Theory*, **11**(2017), no. 2, 393–416.
- [4] A.I.Markushevich, The theory of analytic functions, vol. 2, Moscow, Nauka, 1968 (in Russian).
- [5] E.C.Titchmarsh, The theory of functions, Oxford Univ. Press, Oxford, 1939.

Об одном доказательстве основной теоремы алгебры (многочленов)

Ольга В. Ходос

Институт математики и фундаментальной информатики
Сибирский федеральный университет
Свободный, 79, Красноярск, 660041
Россия

Приводится одно доказательство основной теоремы алгебры (многочленов), основанное на теореме о числе нулей целой функции.

Ключевые слова: основная теорема алгебры, нули целых функций.